§ 7.7 Determinants, Cramer's Rule

The determinant is an important function which associates a number with a square $(n \times n)$ matrix.

First the simple cases.

1×1 matrix $A = [a_{11}]$

$$\text{Det } A = a_{11}$$

2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$\text{Det } A = a_{11}a_{12} - a_{12}a_{21}$$

For larger matrices $(n > 2)$, the determinant is defined recursively. If A is $n \times n$, then Det A is expressed in terms of the determinants of $(n-1) \times (n-1)$ matrices whose determinants in turn are expressed in terms of the determinants of $(n-2) \times (n-2)$ matrices and so on down to 2×2 (or 1×1) matrices whose determinants we know how to calculate.

3.1 Introduction to Determinants

Notation: A_{ij} is the matrix obtained from matrix A by deleting the ith row and jth column of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \qquad A_{23} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Recall that $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$ and we let $\det[a] = a$.

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is given by

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$

Solution

$$\det A = 1\det\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$= \underline{\hspace{6cm}} = \underline{\hspace{1.5cm}}$$

Common notation: $\det\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}$.

So

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix}$$

The **(i, j)-cofactor** of A is the number C_{ij} where $C_{ij} = (-1)^{i+j}\det A_{ij}$.

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 1C_{11} + 2C_{12} + 0C_{13}$$

(cofactor expansion across row 1)

THEOREM 1 The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} \qquad \text{(expansion across row } i\text{)}$$

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \qquad \text{(expansion down column } j\text{)}$$

Use a matrix of signs to determine $(-1)^{i+j}$

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ using cofactor expansion down column 3.

Solution

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = 1.$$

EXAMPLE: Compute the determinant of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{bmatrix}$

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 3 & 5 \end{vmatrix} - 0\begin{vmatrix} 2 & 3 & 4 \\ 2 & 1 & 5 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= 1 \cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 14$$

Method of cofactor expansion is not practical for large matrices - see Numerical Note on page 190.

3.2 Properties of Determinants

THEOREM 3 Let A be a square matrix.

a. If a multiple of one row of A is added to another row of A to produce a matrix B, then $\det A = \det B$.

b. If two rows of A are interchanged to produce B, then $\det B = -\det A$.

c. If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

EXAMPLE: Compute $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 0 & 0 \\ 2 & 7 & 6 & 10 \\ 2 & 9 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 2 & 6 & 10 \\ 2 & 7 & 11 \end{vmatrix} = 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 2 & 7 & 11 \end{vmatrix}$$

$$= 5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -5\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{vmatrix} = \_\_\_\_\_\_\_\_ = \_\_\_\_.$$

Theorem 3(c) indicates that $\begin{vmatrix} * & * & * \\ -2k & 5k & 4k \\ * & * & * \end{vmatrix} = k\begin{vmatrix} * & * & * \\ -2 & 5 & 4 \\ * & * & * \end{vmatrix}$.

Triangular Matrices:

$$\begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \end{bmatrix}$$

(upper triangular) (lower triangular)

THEOREM 2: If A is a triangular matrix, then $\det A$ is the product of the main diagonal entries of A.

EXAMPLE:

$$\begin{vmatrix} 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 4 \end{vmatrix} = \underline{\qquad\qquad\qquad\qquad} = -24$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 & 6 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 7 & 6 & 10 \end{vmatrix} = 2\begin{vmatrix} 1 & 2 & 3 \\ 0 & -4 & -8 \\ 0 & -8 & -11 \end{vmatrix}$$

$$= 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -8 & -11 \end{vmatrix} = 2(-4)\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 2(-4)(1)(1)(5) = -40$$

EXAMPLE: Compute $\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix}$ using a combination of row reduction and cofactor expansion.

Solution

$$\begin{vmatrix} 2 & 3 & 0 & 1 \\ 4 & 7 & 0 & 3 \\ 7 & 9 & -2 & 4 \\ 1 & 2 & 0 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 4 & 7 & 3 \\ 1 & 2 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 4 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= -2\begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & -6 \end{vmatrix} = -2(1)(-1)(-6) = -12.$$

Theorem 3 Further Properties of nth Order Determinants

Let A be an $n \times n$ matrix. Then

a) – c) in Theorem 1 also hold for columns;

d) $\det A^T = \det A$;

e) If one of the rows or colums of A;
is all zeroes, then $\det A = 0$;

f) If one row of A is a scalar multiple of another, then $\det A = 0$.
The same holds for columns.

g) If the rows of A are lin. dep., then $\det A = 0$. The same holds for columns.

Suppose A has been reduced to $U = \begin{bmatrix} \blacksquare & * & * & \cdots & * \\ 0 & \blacksquare & * & \cdots & * \\ 0 & 0 & \blacksquare & \cdots & * \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$ by row replacements and row interchanges, then

$$\det A = \begin{cases} (-1)^r \begin{pmatrix} \text{product of} \\ \text{pivots in } U \end{pmatrix} & \text{when } A \text{ is invertible} \\ 0 & \text{when } A \text{ is not invertible} \end{cases}$$

THEOREM 4 A square matrix is invertible if and only if $\det A \neq 0$.

THEOREM 5 If A is an $n \times n$ matrix, then $\det A^T = \det A$.

Partial proof (2×2 case)

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \qquad \text{and}$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

$$\Rightarrow \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(3 × 3 case)

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$\det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix}$$

$$\Rightarrow \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \det\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

Implications of Theorem 5*?*

Theorem 3 still holds if the word *row* is replaced

with _____________________.

THEOREM 6 (Multiplicative Property)

For $n \times n$ matrices A and B, $\det(AB) = (\det A)(\det B)$.

EXAMPLE: Compute $\det A^3$ if $\det A = 5$.

Solution: $\det A^3 = \det(AAA) = (\det A)(\det A)(\det A)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$

EXAMPLE: For $n \times n$ matrices A and B, show that A is singular if $\det B \neq 0$ and $\det AB = 0$.

Solution: *Since*

$$(\det A)(\det B) = \det AB = 0$$

and

$$\det B \neq 0,$$

then $\det A = 0$. Therefore A is singular.

Theorem 4 Rank in Terms of Determinants

An $m \times n$ matrix $A = (a_{ij})$ has rank $r \geq 1$ iff A has an $r \times r$ submatrix with nonzero determinant, whereas every square submatrix with more than r rows that A has (or does not have!) has determinant equal to 0.

In particular, if A is square, $n \times n$, it has rank n iff

$$\det A \neq 0.$$

Pf.
(Idea). If A has an $r \times r$ submatrix with non-zero det, then the corresponding r rows of A are lin. ind. Thus rank $A = r$

Conversely if A has rank r, let A' be the $r \times r$ submatrix obtained from the pivot rows & columns of A.

Row ops on A $\Longleftrightarrow$ Row ops on A'

and if we reduce A to row echelon form, then A' will reduce to a diagonal matrix with non-zero diagonal entries (which are the r pivots of the row-echelon form of A) and which thus has non-zero det. Also, since rank $A = r$, any larger submatrix will have lin dep. rows & hence 0 det.

Cramer's Rule

- An inefficient way of solving linear systems.

Thm 5 Cramer's Theorem (Solving Linear Systems by Determinants).

a) If a linear system $Ax = b$, or

$$\begin{array}{ccc} a_{11}x_1 + \cdots & + a_{1n}x_n = & b_1 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + \cdots & + a_{nn}x_n = & b_n \end{array}$$

of n eqns in n unknowns has $\operatorname{Det} A \neq 0$ and if we let $a_1, \ldots, a_n$ be the cols. of A, then the system has a unique soln. This soln is given by the formulae

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad \ldots, \quad x_n = \frac{D_n}{D}$$ where

$D = \det A$ and

$$D_i = \det \begin{bmatrix} | & | & & | & & | \\ a_1 & a_2 & \cdots & b_i & \cdots & a_n \\ | & | & & | & & | \end{bmatrix}$$

is the det. of the matrix obtained by replacing the ith col. a_i of A by the vector b.

b) If $Ax = 0$ is homogeneous with $D = \det A \neq 0$, then it has only the trivial soln. (even if $D = 0$, there is always the trivial soln).

Ex. Use Cramer's method to solve

$$Ax = b$$

where $A = \begin{bmatrix} 5 & 3 & 2 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$$D = \det A = \begin{vmatrix} 5 & 3 & 2 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{vmatrix}$$

Cofactor exp on first col.

$$= 5(15-8) - 2(9+8) + 0$$

$$= 5 \times 7 - 2 \times 17$$

$$= 35 - 34 = 1$$

$$D_1 = \det [b\ a_2\ a_3] = \begin{vmatrix} 1 & 3 & 2 \\ -2 & 5 & -2 \\ 3 & -4 & 3 \end{vmatrix}$$

$$= 1(15-8) - (-2)(9+8) + 3(-6-10)$$

$$= 7 + 2 \times 17 + 3 \times -16$$

$$= 7 + 34 - 48 = -7$$

$$D_2 = \det \begin{bmatrix} a_1 & b & a_2 \end{bmatrix} = \begin{vmatrix} 5 & 1 & 2 \\ 2 & -2 & -2 \\ 0 & 3 & 3 \end{vmatrix}$$

$$= 5(-6+6) - 2(3-6) + 0$$

$$= 0 - 2(-3)$$

$$= 6$$

$$D_3 = \det \begin{bmatrix} a_1 & a_2 & b \end{bmatrix} = \begin{vmatrix} 5 & 3 & 1 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{vmatrix}$$

$$= 5(15-8) - 2(9+4) + 0$$

$$= 5 \times 7 - 2 \times 13$$

$$= 35 - 26$$

$$= 9$$

So $x_1 = \dfrac{D_1}{D} = \dfrac{-7}{1} = -7$

$$x_2 = \frac{D_2}{D} = \frac{6}{1} = 6$$

$$x_3 = \frac{D_3}{D} = \frac{9}{1} = 9$$

Ans. $x = \begin{bmatrix} -7 \\ 6 \\ 9 \end{bmatrix}$

Check.

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & 5 & -2 \\ 0 & -4 & 3 \end{bmatrix}\begin{bmatrix} -7 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} -35+18+18 \\ -14+30-18 \\ 0 \ -24+27 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \checkmark$$