§4.3 Constant - Coefficient Systems

We consider homog. linear systems

$$\underline{y}' = A\underline{y}$$

where the entries in the $n \times n$ coeff. matrix $A = [a_{jk}]$ are <u>constants</u> (do not depend on t).

<u>Why we Bothered with Eigenvalues / Eigenvectors</u>

Suppose we try for a sol^n^ of this system of the form

$$\underline{y} = \underline{x}e^{\lambda t}$$

for some vector $\underline{x}$ in $\mathbb{R}^n$.

Then $\underline{y}' = \lambda \underline{x} e^{\lambda t}$ and if

we have a sol$^{\underline{n}}$ iff.

$$\underline{y}' = A\underline{y}$$

ie $\lambda \underline{x} e^{\lambda t} = A \underline{x} e^{\lambda t}$

and since $e^{\lambda t} \neq 0$ this holds iff

$$\lambda \underline{x} = A\underline{x}$$

or $A\underline{x} = \lambda \underline{x}$.

ie λ is an e-value of A and

$\underline{x}$ is an e-vector for this e-value.

<u>Moral</u> $\underline{y} = \underline{x} e^{\lambda t}$ is a sol^n of

$$\underline{y}' = A \underline{y}$$

iff

λ is an e-value of A &

$\underline{x}$ is an e-vector with e-value λ.

Recall that for an $n \times n$ matrix, the char eq^n is a poly. of degree n with roots $\lambda_1, \ldots, \lambda_n$, say (some may be the same if we have multiple roots).

Suppose now we have n sol^{n}s

$$\underline{y}^{(1)} = \underline{x}^{(1)} e^{\lambda_1 t}, \quad \ldots \quad , \underline{y}^{(n)} = \underline{x}^{(n)} e^{\lambda_n t}$$

and we want to know if we have a basis of sol^{n}s.

To see if these sol^{ns}s are lin ind., we take the Wronskian

$$W = W(\underline{Y}^{(1)}, \dots, \underline{Y}^{(n)}) = \begin{vmatrix} x_1^{(1)} e^{\lambda_1 t} & \cdots & x_1^{(n)} e^{\lambda_n t} \\ \vdots & & \vdots \\ x_n^{(n)} e^{\lambda_1 t} & \cdots & x_n^{(n)} e^{\lambda_n t} \end{vmatrix}$$

$$= e^{(\lambda_1 + \dots + \lambda_n)t} \begin{vmatrix} x_1^{(1)} & \cdots & x_1^{(n)} \\ \vdots & & \vdots \\ x_n^{(1)} & & x_n^{(n)} \end{vmatrix}.$$

If we now remember that for an $n \times n$ matrix B, $\det B \neq 0$ iff the columns of B are lin ind., then we have

$\underline{Y}^{(1)}, \ldots, \underline{Y}^{(n)}$ are a basis of sol^ns of

$$\underline{Y}' = A\underline{Y}$$

iff

$\underline{X}^{(1)}, \ldots, \underline{X}^{(n)}$ is a lin ind set of e-vectors of A, ie. a basis of $\mathbb{R}^n$ consisting of e-vectors of A.

We have proved

<u>Thm 1</u> General Solution.

If the matrix A in the system $\underline{Y}' = A\underline{Y}$ has a basis of e-vectors of $\mathbb{R}^n$ (ie n lin ind e-vectors), then the corresponding sol^ns $\underline{Y}^{(1)}, \ldots, \underline{Y}^{(n)}$ are lin ind as f^ns (on $\mathbb{R}$) and a general sol^n is given by

$$\underline{Y} = c_1\underline{Y}^{(1)} + \cdots + c_n\underline{Y}^{(n)}.$$

N.b. For 'most' matrices, we can find a basis of e-vectors. (e.g. Symmetric $A^T = A$
Skew-Symmetric $A^T = -A$)

Even when we cannot do this, we can still say something (Jordan canonical form).

Graphing Solutions in the Phase Plane

Consider systems of 2 linear ODEs with constant coeffs.

$$\underline{y}' = A\underline{y}$$

ie

$$y_1' = a_{11} y_1 + a_{12} y_2$$

$$y_2' = a_{21} y_1 + a_{22} y_2 .$$

We can graph $\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, in the usual way (in $\mathbb{R}^3$), or we can graph $\underline{y}(t)$ as a parametric curve in the $y_1 y_2$-plane which is called the phase plane.

The curve $\underline{y}(t)$ in the phase plane is called a trajectory or orbit. Several such curves which indicate the behaviour of the system are called a phase portrait.

Note that if we set $\underline{y} = 0$, then

$\underline{y}' = 0$ and $A\underline{y} = 0$ also, so we

have a constant solution.

$\underline{y} = 0$ is known as a critical point

or equilibrium point.

In a linear system with const. coeffs, there can only be one crit pt. In general there can be more than 1 such point.

Note that by calculus

$$\frac{dy_2}{dy_1} = \frac{y_2' \, dt}{y_1' \, dt} = \frac{y_2'}{y_1'} = \frac{a_{21} y_1 + a_{22} y_2}{a_{11} y_1 + a_{12} y_2}$$

and as $y_1, y_2 \to 0$, $\frac{dy_2}{dy_1}$ tends to $\frac{0}{0}$

which is undefined.

Ex 1

$$\underline{y}' = A\underline{y} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \underline{y}$$

or $\quad y_1 = -3y_1 + y_2$

$\quad\quad y_2 = y_1 - 3y_2 .$

Sol^n. Set $\underline{y} = \underline{x}\, e^{\lambda t}$ & subst into the system to get $A\underline{x} = \lambda \underline{x}$.

Char eq^n of A is

$$\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & 1 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 8 = 0$$

This can be factorized into

$$(\lambda + 2)(\lambda + 4) = 0$$

which gives e-values $\lambda_1 = -2, \ \lambda_2 = -4$.

For $\lambda_1 = -2$, to find the e-vector we have the linear system

$$\begin{bmatrix} -3-(-2) & 1 \\ 1 & -3(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and a sol^{n} is clearly $\underline{X}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as an e-vector.

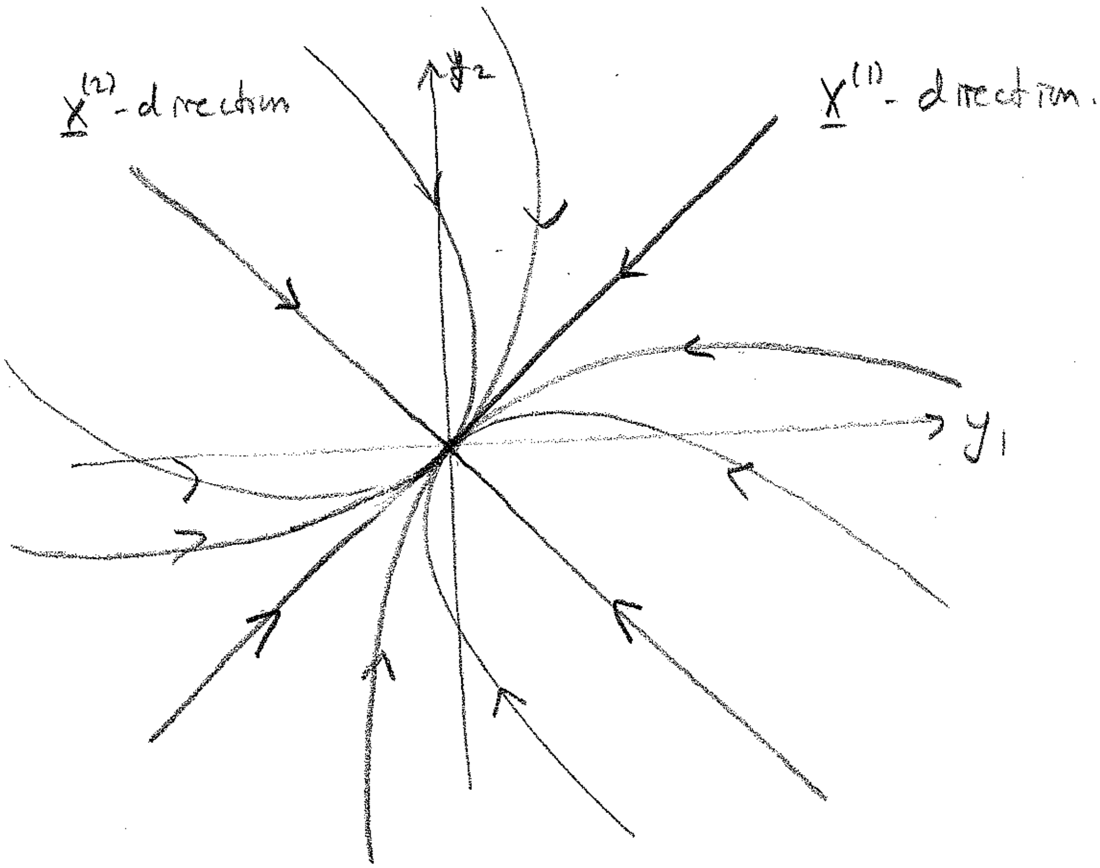
Similarly, for $\lambda_2 = -4$, we find that

$\underline{X}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an e-vector.

This gives the general sol^{n}

$$\underline{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \underline{Y}^{(1)} + c_2 \underline{Y}^{(2)} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$$

Phase portrait looks like



Stable, proper <u>node</u>.

(9) becomes 0/0. This point P_0 ...
point of (6).

Five Types of Critical Points

There are five types of critical points depending on the geometric shape of the trajectories near them. They are called **improper nodes, proper nodes, saddle points, centers,** and **spiral points.** We define and illustrate them in Examples 1–5.

EXAMPLE 1 ***(Continued)* Improper Node (Fig. 81)**

An **improper node** is a critical point P_0 at which all the trajectories, except for two of them, have the same limiting direction of the tangent. The two exceptional trajectories also have a limiting direction of the tangent at P_0 which, however, is different.

The system (8) has an improper node at **0**, as its phase portrait Fig. 81 shows. The common limiting direction at **0** is that of the eigenvector $\mathbf{x}^{(1)} = [1 \quad 1]^\mathsf{T}$ because e^{-4t} goes to zero faster than e^{-2t} as t increases. The two exceptional limiting tangent directions are those of $\mathbf{x}^{(2)} = [1 \quad -1]^\mathsf{T}$ and $-\mathbf{x}^{(2)} = [-1 \quad 1]^\mathsf{T}$. ■

EXAMPLE 2 **Proper Node (Fig. 82)**

A **proper node** is a critical point P_0 at which every trajectory has a definite limiting direction and for any given direction **d** at P_0 there is a trajectory having **d** as its limiting direction.

The system

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= y_2 \end{aligned} \tag{10}$$

has a proper node at the origin (see Fig. 82). Indeed, the matrix is the unit matrix. Its characteristic equation $(1-\lambda)^2 = 0$ has the root $\lambda = 1$. Any $\mathbf{x} \neq \mathbf{0}$ is an eigenvector, and we can take $[1 \quad 0]^\mathsf{T}$ and $[0 \quad 1]^\mathsf{T}$. Hence a general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^t \end{aligned} \qquad \text{or} \qquad c_1 y_2 = c_2 y_1.$$ ■

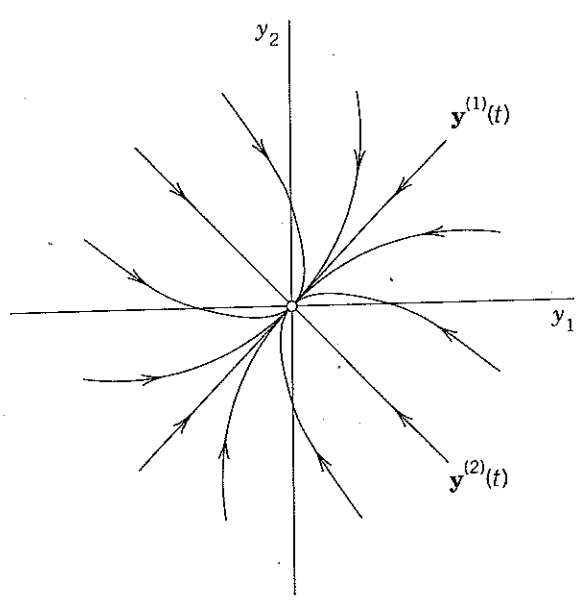


Fig. 81. Trajectories of the system (8) (Improper node)

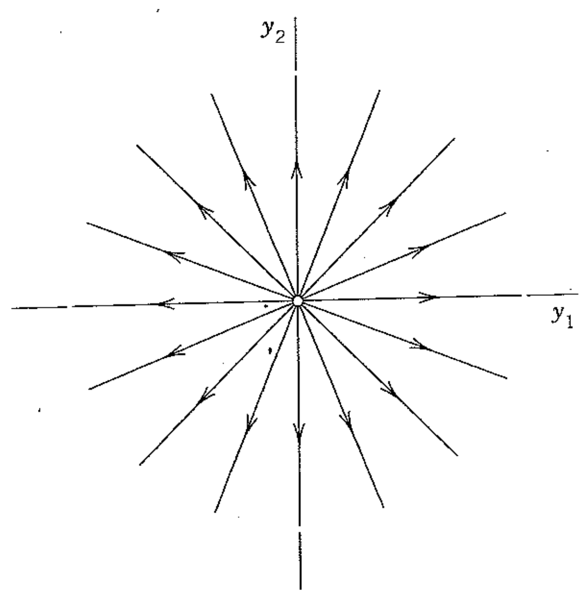


Fig. 82. Trajectories of the system (10) (Proper node)

EXAMPLE 3 **Saddle Point (Fig. 83)**

A **saddle point** is a critical point P_0 at which there are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of P_0 bypass P_0.

The system

$$(11) \qquad \mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_1 \\ y_2' &= -y_2 \end{aligned}$$

has a saddle point at the origin. Its characteristic equation $(1 - \lambda)(-1 - \lambda) = 0$ has the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. For $\lambda = 1$ an eigenvector $[1 \quad 0]^\mathsf{T}$ is obtained from the second row of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, that is, $0x_1 + (-1 - 1)x_2 = 0$. For $\lambda_2 = -1$ the first row gives $[0 \quad 1]^\mathsf{T}$. Hence a general solution is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-t} \qquad \text{or} \qquad \begin{aligned} y_1 &= c_1 e^t \\ y_2 &= c_2 e^{-t} \end{aligned} \qquad \text{or} \qquad y_1 y_2 = const.$$

This is a family of hyperbolas (and the coordinate axes); see Fig. 83. ■

EXAMPLE 4 **Center (Fig. 84)**

A **center** is a critical point that is enclosed by infinitely many closed trajectories.

The system

$$(12) \qquad \mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= y_2 \\ y_2' &= -4y_1 \end{aligned}$$

has a center at the origin. The characteristic equation $\lambda^2 + 4 = 0$ gives the eigenvalues $2i$ and $-2i$. For $2i$ an eigenvector follows from the first equation $-2ix_1 + x_2 = 0$ of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, say, $[1 \quad 2i]^\mathsf{T}$. For $\lambda = -2i$ that equation is $-(-2i)x_1 + x_2 = 0$ and gives, say, $[1 \quad -2i]^\mathsf{T}$. Hence a complex general solution is

$$(12^*) \qquad \mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} + c_2 \begin{bmatrix} 1 \\ -2i \end{bmatrix} e^{-2it}, \qquad \text{thus} \qquad \begin{aligned} y_1 &= \quad c_1 e^{2it} + \quad c_2 e^{-2it} \\ y_2 &= 2ic_1 e^{2it} - 2ic_2 e^{-2it}. \end{aligned}$$

The next step would be the transformation of this solution to real form by the Euler formula (Sec. 2.2). But we were just curious to see what kind of eigenvalues we obtain in the case of a center. Accordingly, we do not continue, but start again from the beginning and use a shortcut. We rewrite the given equations in the form $y_1' = y_2$, $4y_1 = -y_2'$; then the product of the left sides must equal the product of the right sides,

$$4y_1 y_1' = -y_2 y_2'. \qquad \text{By integration,} \qquad 2y_1^2 + \tfrac{1}{2}y_2^2 = const.$$

This is a family of ellipses (see Fig. 84) enclosing the center at the origin. ■

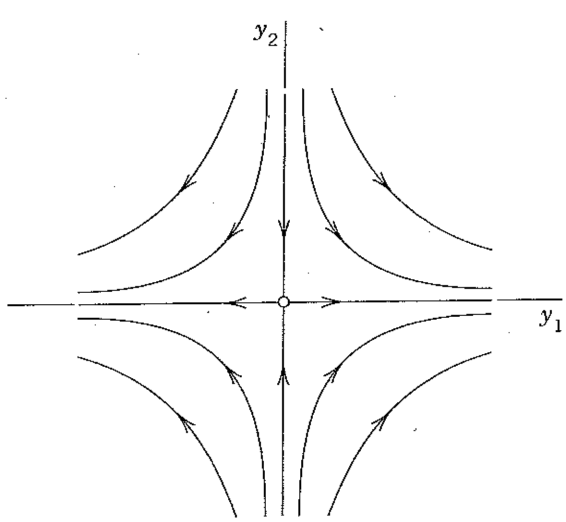


Fig. 83. Trajectories of the system (11) (Saddle point)

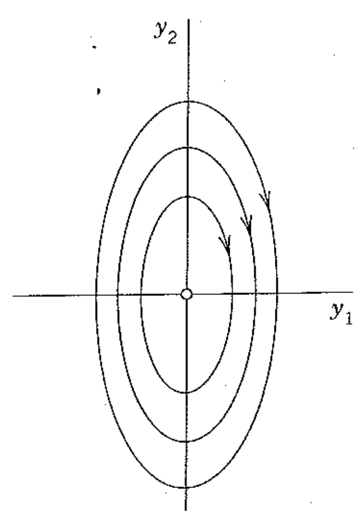


Fig. 84. Trajectories of the system (12) (Center)

EXAMPLE 5 Spiral Point (Fig. 85)

A **spiral point** is a critical point P_0 about which the trajectories spiral, approaching P_0 as $t \to \infty$ (or tracing these spirals in the opposite sense, away from P_0).

The system

$$\text{(13)} \qquad \mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{y}, \qquad \text{thus} \qquad \begin{aligned} y_1' &= -y_1 + y_2 \\ y_2' &= -y_1 - y_2 \end{aligned}$$

has a spiral point at the origin, as we shall see. The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. It gives the eigenvalues $-1 + i$ and $-1 - i$. Corresponding eigenvectors are obtained from $(-1 - \lambda)x_1 + x_2 = 0$. For $\lambda = -1 + i$ this becomes $-ix_1 + x_2 = 0$ and we can take $[1 \quad i]^{\mathsf{T}}$ as an eigenvector. Similarly, an eigenvector corresponding to $-1 - i$ is $[1 \quad -i]^{\mathsf{T}}$. This gives the complex general solution

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-1-i)t}.$$

The next step would be the transformation of this complex solution to a real general solution by the Euler formula. But, as in the last example, we just wanted to see what eigenvalues to expect in the case of a spiral point. Accordingly, we start again from the beginning and instead of that rather lengthy systematic calculation we use a shortcut. We multiply the first equation in (13) by y_1, the second by y_2, and add, obtaining

$$y_1y_1' + y_2y_2' = -({y_1}^2 + {y_2}^2).$$

We now introduce polar coordinates r, t, where $r^2 = {y_1}^2 + {y_2}^2$. Differentiating this with respect to t gives $2rr' = 2y_1y_1' + 2y_2y_2'$. Hence the previous equation can be written

$$rr' = -r^2, \qquad \text{Thus,} \qquad r' = -r, \qquad dr/r = -dt, \qquad \ln|r| = -t + c^*, \qquad r = ce^{-t}.$$

For each real c this is a spiral, as claimed. (see Fig. 85). ■

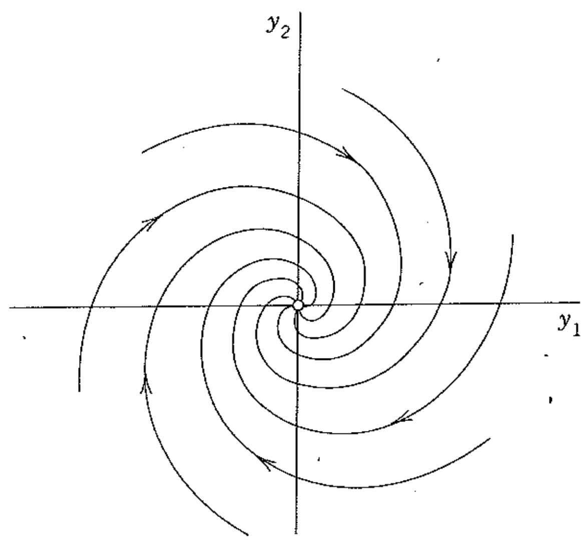


Fig. 85. Trajectories of the system (13) (Spiral point)

EXAMPLE 6 No Basis of Eigenvectors Available. Degenerate Node (Fig. 86)

This cannot happen if $\mathbf{A}$ in (1) is symmetric ($a_{kj} = a_{jk}$, as in Examples 1–3) or skew-symmetric ($a_{kj} = -a_{jk}$, thus $a_{jj} = 0$). And it does not happen in many other cases (see Examples 4 and 5). Hence it suffices to explain the method to be used by an example.

Find and graph a general solution of

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \mathbf{y}. \tag{14}$$

Solution. $\mathbf{A}$ is not skew-symmetric! Its characteristic equation is

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.$$

It has a double root $\lambda = 3$. Hence eigenvectors are obtained from $(4 - \lambda)x_1 + x_2 = 0$, thus from $x_1 + x_2 = 0$, say, $\mathbf{x}^{(1)} = [1 \quad -1]^\mathsf{T}$ and nonzero multiples of it (which do not help). The method now is to substitute

$$\mathbf{y}^{(2)} = \mathbf{x}te^{\lambda t} + \mathbf{u}e^{\lambda t}$$

with constant $\mathbf{u} = [u_1 \quad u_2]^\mathsf{T}$ into (14). (The $\mathbf{x}t$-term alone, the analog of what we did in Sec. 2.2 in the case of a double root, would not be enough. Try it.) This gives

$$\mathbf{y}^{(2)\prime} = \mathbf{x}e^{\lambda t} + \lambda \mathbf{x}te^{\lambda t} + \lambda \mathbf{u}e^{\lambda t} = \mathbf{A}\mathbf{y}^{(2)} = \mathbf{A}\mathbf{x}te^{\lambda t} + \mathbf{A}\mathbf{u}e^{\lambda t}.$$

On the right, $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Hence the terms $\lambda\mathbf{x}te^{\lambda t}$ cancel, and then division by $e^{\lambda t}$ gives

$$\mathbf{x} + \lambda\mathbf{u} = \mathbf{A}\mathbf{u}, \qquad \text{thus} \qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{x}.$$

Here $\lambda = 3$ and $\mathbf{x} = [1 \quad -1]^\mathsf{T}$, so that

$$(\mathbf{A} - 3\mathbf{I})\mathbf{u} = \begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \text{thus} \qquad \begin{aligned} u_1 + u_2 &= 1 \\ -u_1 - u_2 &= -1. \end{aligned}$$

A solution, linearly independent of $\mathbf{x} = [1 \quad -1]^\mathsf{T}$, is $\mathbf{u} = [0 \quad 1]^\mathsf{T}$. This yields the answer (Fig. 86)

$$\mathbf{y} = c_1\mathbf{y}^{(1)} + c_2\mathbf{y}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{3t}.$$

The critical point at the origin is often called a **degenerate node.** $c_1\mathbf{y}^{(1)}$ gives the heavy straight line, with $c_1 > 0$ the lower part and $c_1 < 0$ the upper part of it. $\mathbf{y}^{(2)}$ gives the right part of the heavy curve from 0 through the second, first, and—finally—fourth quadrants. $-\mathbf{y}^{(2)}$ gives the other part of that curve. ∎

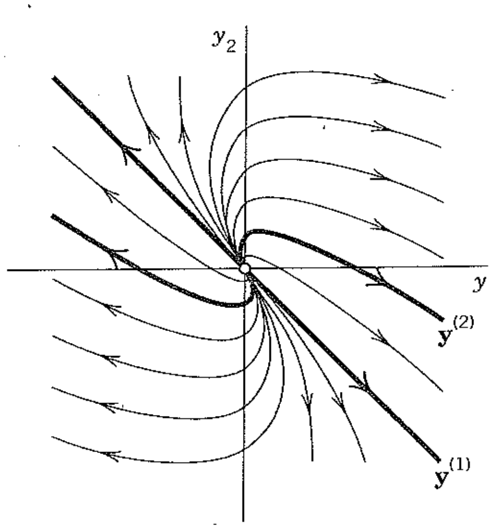


Fig. 86. Degenerate node in Example 6