§ 2.7 Inhomogeneous ODEs

In this section we discuss solutions of second order linear inhomogeous ODEs

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

with $r(x) \not\equiv 0$ and show how they relate to sol$^{\text{n}}$s of the associated homogeneous ODE

(2) $$y'' + p(x)y' + q(x)y = 0.$$

Defn A general soln of the inhomog. ODE

(1) on an open interval I is a soln

of the form

(3) $y(x) = y_h(x) + y_p(x)$;

here $y_h = c_1 y_1 + c_2 y_2$ is a general soln

of the associated homog. ODE (2) on I

and y_p is any soln of (1) on I

containing no arb. consts.

We need to do 2 things:

1. show these defns make sense;

2. find y_p.

Thm 1 Relation of Sol$^{\underline{n}}$s of the Inhomog ODE to Those of the Associated Homog. ODE.

(a) The sum of a sol$^{\underline{n}}$ y of (1) on some open interval I and a sol$^{\underline{n}}$ $\tilde{y}$ of (2) on I is another sol$^{\underline{n}}$ of (1) on I. In particular, (3) is a sol$^{\underline{n}}$ of (1) on I.

(b) The difference of two sol$^{\underline{n}}$s of (1) on I is a sol$^{\underline{n}}$ of (2) on I.

Pf. (a) Let y be a sol$^{\underline{n}}$ of (1) and $\tilde{y}$ a sol$^{\underline{n}}$ of (2) on I.

Then $(y+\tilde{y})'' + p(x)(y+\tilde{y})' + q(x)(y+\tilde{y})$

$= (y'' + p(x)y' + q(x)y) + (\tilde{y}'' + p(x)\tilde{y}' + q(x)\tilde{y})$

$= r(x) + 0 = r(x)$ as required.

(b) Now let y, y^* be two solns of (1) on I.

Then

$$(y-y^*)'' + p(x)(y-y^*)' + q(x)(y-y^*)$$

$$= (y'' + p(x)y' + q(x)y) - (y^{*''} + p(x)y^{*'} + q(x)y^*)$$

$$= r(x) - r(x)$$

$$= 0$$ as required. ▨

We already know that for homog. ODEs, the general soln includes all solns. We show the same is true for inhomog. ODEs.

<u>Theorem 2</u> A General Solution of an Inhomogeneous ODE Includes all Solutions.

If $p(x), q(x), r(x)$ in (1) are cts on some open interval I, then every soln of (1) on I is obtained by assigning suitable values to the constants c_1, c_2 in a general soln (3) of (1) on I.

Pf. Let y^* be any sol^n of (1) on I and let x_0 be any pt. of I. Let (3) be any general sol^n of (1) on I.

Note that this solution exists.

y_h exists by Thm 3 in § 2.6

while we will show in § 2.10 that y_p exists also.

By Thm 1 (b), the difference $Y = y^* - y_p$ is a sol^n of the homog eq^n (2) on I.

At x_0 we have

$$Y(x_0) = y^*(x_0) - y_p(x_0)$$
$$Y'(x_0) = y^{*\prime}(x_0) - y_p'(x_0).$$

By Thm 1 in § 2.6 we can find a unique particular sol^n Y satisfying these initial conditions (as for any others), by choosing our consts c_1, c_2 suitably.

Then, on I

$$y^* = Y + y_p$$

$$= c_1 y_1 + c_2 y_2 + y_p$$

$\blacksquare$

as required.

Table for Undetermined Coefficients

Term in $r(x)$	Choice for $y_p(x)$
$kx^n, \; n = 0, 1, 2, \ldots$	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$k\cos(\omega x)$ $k\sin(\omega x)$	$K\cos(\omega x) + M\sin(\omega x)$
$ke^{\alpha x}\cos(\omega x)$ $ke^{\alpha x}\sin(\omega x)$	$e^{\alpha x}\left(K\cos(\omega x) + M\sin(\omega x)\right)$.

In addition to the table, we also need the following rules.

Choice Rules for Undetermined Coefficients.

(a) <u>Basic Rule</u> If $r(x)$ in (4) is one of the fns in the first column of the table choose y_p from the same line of the table and determine its undetermined coeffs. by substituting y_p and its derivatives into (4).

(b) <u>Modification Rule</u> If a term in your choice for y_p happens to be a soln of the homog. ODE corresponding to (4), multiply your choice of y_p by x (or x^2 if this soln corresponds to a double root of the char eqn of the homog ODE).

(c) If $r(x)$ is a sum of fns in the first column of the table, choose for y_p the sum of the fns in the same lines on the rhs. of the table.

Method of Undetermined Coefficients

From earlier we see that to solve the inhomog. ODE (1), we need to solve the associated homog ODE (2) and find any sol$^{\underline{n}}$ y_p of (1) so that we get the general sol$^{\underline{n}}$ (3) of (1).

How to find y_p?

We will discuss two methods, <u>undetermined coefficients</u> and <u>variation of parameters</u> (in § 2.10).

Variation of parameters is more complex but more general.

Undetermined coefficients is simpler but less general.

Undetermined coefficients is suitable for linear ODEs with <u>constant coefficients</u> a, b

$$(4) \qquad y'' + ay' + by = r(x)$$

and where $r(x)$ is either

1. a polynomial
2. an exponential fn.
3. a sine or cosine fn.
4. a sum or product of fns. from 1., 2., 3.

There are certain basic rules for choosing y_p based on $r(x)$ (and also a, b).

Ex 1 Application of the Basic Rule (a).

Solve the IVP

$$y'' + y = 0.001x^2, \qquad y(0) = 0, \; y'(0) = 1.5$$

Soln. Step 1. General Soln of the Associated Homog. ODE.

The ODE $y'' + y = 0$ has the general soln.

$$y_h = A\cos x + B\sin x.$$

Step 2. Soln y_p of the Inhomog. ODE

First try $y_p = Kx^2$ (simplest possible choice).

Then $y_p' = 2Kx$, $y_p'' = 2K$ and if we subst in the ODE we get

$$2K + Kx^2 = .001x^2$$

For this to hold for all x, the coeffs of the different powers of x must match. Thus

$\underline{x^0}$ $\quad 2K = 0$

$\underline{x^2}$ $\quad K = .001$

which is clearly impossible.

Suppose now instead, we try $y_p = K_2 x^2 + K_1 x + K_0$ (again using the table). Then

$$y_p' = 2K_2 x + K_1, \quad y_p'' = 2K_2 \quad \text{and so}$$

$$2K_2 + K_2 x^2 + K_1 x + K_0 = .001x^2.$$

$$K_2 x^2 + K_1 x + (K_0 + 2K_2) = .001x^2.$$

Again the powers of x must match on both sides.

$$x^2: \quad K_2 = .001$$

$$x \quad K_1 = 0$$

$$x^0 \quad K_0 + 2K_2 = 0$$

Thus $K_0 = -2K_2 = -.002, \; K_1 = 0, \; K_2 = .001$

and so

$$y_p = .0001x^2 - .0002$$

and

$$y = y_h + y_p = A\cos x + B\sin x + .001x^2 - .002$$

Step 3. Solution of the initial value problem. Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

$$y' = y_h' + y_p' = -A \sin x + B \cos x + 0.002x \qquad \text{and} \qquad y'(0) = B = 1.5.$$

This gives the answer (Fig. 49)

$$y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.$$

Figure 49 shows y as well as the quadratic parabola y_p about which y is oscillating, practically like a sine curve since the cosine term is smaller by a factor of about 1/1000. ■

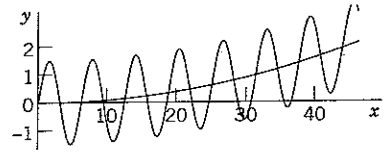

Fig. 49. Solution in Example 1

EXAMPLE 2 Application of the Modification Rule (b)

Solve the initial value problem

$$(6) \qquad y'' + 3y' + 2.25y = -10\, e^{-1.5x}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$. Hence the homogeneous ODE has the general solution

$$y_h = (c_1 + c_2 x)e^{-1.5x}.$$

Step 2. Solution y_p of the nonhomogeneous ODE. The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But we see from y_h that this function is a solution of the homogeneous ODE, which corresponds to a *double root* of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by x^2. That is, we choose

$$y_p = Cx^2e^{-1.5x}. \qquad \text{Then} \qquad y_p' = C(2x - 1.5x^2)e^{-1.5x}, \qquad y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}.$$

We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10.$$

Comparing the coefficients of x^2, x, x^0 gives $0 = 0$, $0 = 0$, $2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^2e^{-1.5x}$. Hence the given ODE has the general solution

$$y = y_h + y_p = (c_1 + c_2x)e^{-1.5x} - 5x^2e^{-1.5x}.$$

Step 3. Solution of the initial value problem. Setting $x = 0$ in y and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of y gives

$$y' = (c_2 - 1.5c_1 - 1.5c_2x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2e^{-1.5x}.$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer (Fig. 50)

$$y = (1 + 1.5x)\, e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x}.$$

The curve begins with a horizontal tangent, crosses the x-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as x increases. ■

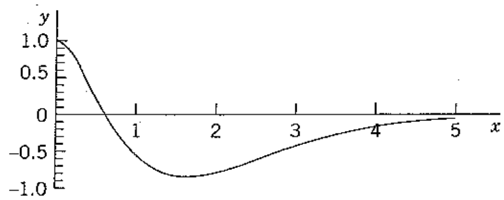

Fig. 50. Solution in Example 2

EXAMPLE 3 Application of the Sum Rule (c)

Solve the initial value problem

$$(7) \qquad y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x, \qquad y(0) = 0.16, \qquad y'(0) = 40.08.$$

Solution. ***Step 1. General solution of the homogeneous ODE.*** The characteristic equation

$$\lambda^2 + 2\lambda + 5 = (\lambda + 1 + 2i)(\lambda + 1 - 2i) = 0$$

shows that a real general solution of the homogeneous ODE is

$$y_h = e^{-x}(A\cos 2x + B\sin 2x).$$

Step 2. Solution of the nonhomogeneous ODE. We write $y_p = y_{p1} + y_{p2}$, where y_{p1} corresponds to the exponential term and y_{p2} to the sum of the other two terms. We set

$$y_{p1} = Ce^{0.5x}. \qquad \text{Then} \qquad y'_{p1} = 0.5Ce^{0.5x} \qquad \text{and} \qquad y''_{p1} = 0.25Ce^{0.5x}.$$

Substitution into the given ODE and omission of the exponential factor gives $(0.25 + 2 \cdot 0.5 + 5)C = 1$, hence $C = 1/6.25 = 0.16$, and $y_{p1} = 0.16e^{0.5x}$.

We now set $y_{p2} = K\cos 10x + M\sin 10x$, as in Table 2.1, and obtain

$$y'_{p2} = -10K\sin 10x + 10M\cos 10x, \qquad y''_{p2} = -100K\cos 10x - 100M\sin 10x.$$

Substitution into the given ODE gives for the cosine terms and for the sine terms

$$-100K + 2 \cdot 10M + 5K = 40, \qquad -100M - 2 \cdot 10K + 5M = -190$$

or, by simplification,

$$-95K + 20M = 40, \qquad -20K - 95M = -190.$$

The solution is $K = 0$, $M = 2$. Hence $y_{p2} = 2\sin 10x$. Together,

$$y = y_h + y_{p1} + y_{p2} = e^{-x}(A\cos 2x + B\sin 2x) + 0.16e^{0.5x} + 2\sin 10x.$$

Step 3. Solution of the initial value problem. From y and the first initial condition, $y(0) = A + 0.16 = 0.16$, hence $A = 0$. Differentiation gives

$$y' = e^{-x}(-A\cos 2x - B\sin 2x - 2A\sin 2x + 2B\cos 2x) + 0.08e^{0.5x} + 20\cos 10x.$$

From this and the second initial condition we have $y'(0) = -A + 2B + 0.08 + 20 = 40.08$, hence $B = 10$. This gives the solution (Fig. 51)

$$y = 10e^{-x}\sin 2x + 0.16e^{0.5x} + 2\sin 10x.$$

The first term goes to 0 relatively fast. When $x = 4$, it is practically 0, as the dashed curves $\pm 10e^{-x} + 0.16e^{0.5x}$ show. From then on, the last term, $2\sin 10x$, gives an oscillation about $0.16e^{0.5x}$, the monotone increasing dashed curve. ■

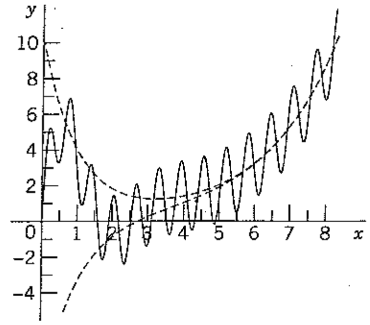

Fig. 51. Solution in Example 3

Stability. The following is important. If (and only if) all the roots of the characteristic equation of the homogeneous ODE $y'' + ay' + by = 0$ in (4) are negative, or have a negative real part, then a general solution y_h of this ODE goes to 0 as $x \to \infty$, so that the **"transient solution"** $y = y_h + y_p$ of (4) approaches the **"steady-state solution"** y_p. In this case the nonhomogeneous ODE and the physical or other system modeled by the ODE are called **stable;** otherwise they are called **unstable.** For instance, the ODE in Example 1 is unstable.

Basic applications follow in the next two sections.