§ 13.5 The Exponential Function

Idea: Extend the usual exponential fn e^x from calculus to the complex numbers.

For $z = x + iy \in \mathbb{C}$ define the exponential of z, e^z or $\exp z$ by

$$e^z := e^x(\cos y + i \sin y).$$

Note that if $z = x \in \mathbb{R}$, then $y = 0$ so $\cos y = 1$, $\sin y = 0$ and

$$e^z = e^x,$$

so our new definition does indeed extend e^x.

If $z = iy$, $x = 0$ and

$e^z = \cos y + i\sin y$, so

$$\boxed{e^{iy} = \cos y + i\sin y}$$

Hence for a cplx no $z = r(\cos\theta + i\sin\theta)$ in polar form, we may write

$$\boxed{z = re^{i\theta}}$$

Special cases of the above are

$\boxed{e^{i\pi} = -1}$, Euler's formula

$e^{2\pi i} = 1$, $e^{i\frac{\pi}{2}} = i$, $e^{-i\frac{\pi}{2}} = -i$

Note that since

$$e^z = e^x(\cos y + i\sin y)$$

is already in polar form, we see that

$$|e^z| = e^x \quad , \quad \arg(e^z) = y$$ up to integer multiples of 2π.

In particular, $|e^{iy}| = 1$ $(x=0)$.

Also

$$e^{z+2\pi i} = e^x(\cos(y+2\pi) + i\sin(y+2\pi))$$

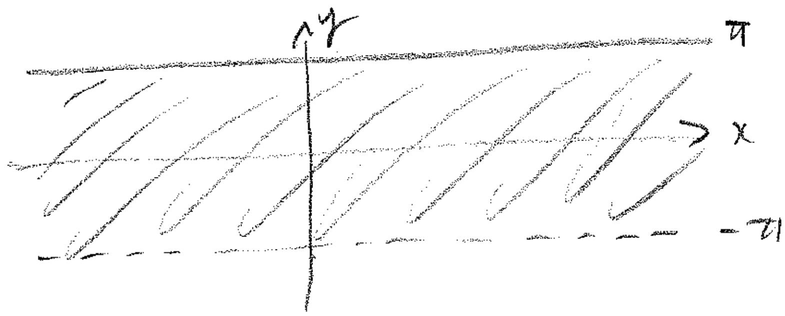
$$= e^x(\cos y + i\sin y)$$ by periodicity of cos, sin

$$= e^z.$$

So $e^{z+2\pi i} = e^z$. Say e^z is <u>periodic</u> with <u>period</u> 2π.

Thus e^z will assume all its possible values on a horizontal strip of width 2π, e.g. the strip

$$S = \{z = x+iy : -\pi < y \leq \pi\}.$$



Finally note that since $|e^z| = e^x$ and $e^x \neq 0$ for all $x \in \mathbb{R}$, e^z is never 0 in value. Say e^z is entire and non-vanishing.

If $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, then

$$e^{z_1} e^{z_2} = e^{x_1 + iy_1} e^{x_2 + iy_2}$$

$$= e^{x_1} (\cos y_1 + i \sin y_1) \cdot e^{x_2} (\cos y_2 + i \sin y_2)$$

$$= e^{x_1} e^{x_2} \left(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i \left(\begin{array}{l} \cos y_1 \sin y_2 \\ + \sin y_1 \cos y_2 \end{array} \right) \right)$$

$$= e^{x_1 + x_2} (\cos(y_1 + y_2) + i \sin(y_1 + y_2))$$

by compound angle & $e^a e^b = e^{a+b}$

$$= e^{x_1 + x_2 + i(y_1 + y_2)}$$

$$= e^{z_1 + z_2}.$$

So $\boxed{e^{z_1 + z_2} = e^{z_1} e^{z_2}}$ (like $e^{a+b} = e^a e^b$ in calculus)

Note if $z = x + iy$, then

$$e^z = e^x e^{iy}.$$

We can define the derivative of a complex fn by analogy with the derivative from calculus

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad (h \text{ is } \underline{\text{complex}}).$$

With this defn one can show e^z is differentiable everywhere and

$$\frac{d}{dz}(e^z) = e^z \quad , \; z \in \mathbb{C}$$

just like $\frac{d}{dx}(e^x) = e^x, \quad x \in \mathbb{R}$.

Fns which are differentiable in this sense are often called analytic or holomorphic. Fns like e^z which are analytic everywhere are called <u>entire</u> fns.