§ 1.2 Geometric Meaning of $y' = f(x,y)$

Direction Fields

Consider a first order ODE of the type

$$y' = f(x,y)$$

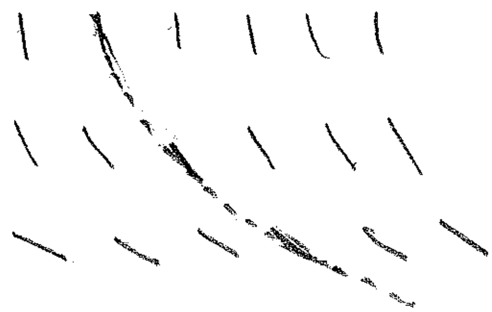
At a given pt (x_0, y_0),

$$\frac{dy}{dx} = f(x_0, y_0)$$

gives us the slope of the sol^{n} $y(x)$ passing through (x_0, y_0).

This leads to an idea for drawing solution curves for the ODE.

Idea. Draw little lines with slope $f(x_0, y_0)$ at many different points and then 'join the lines'.



Has 2 advantages.

1. Enables us to draw solutions without having to actually solve the ODE
2. Shows us how the solutions behave graphically (qualitatively).

Has 3 disadvantes.

1. The method gives only approximate, numerical solutions.
2. Can't analyze the solutions in a precise mathematical way.
3. It's tedious!

One way around 3 is to use a computer algebra system (CAS) such as Mathematica or Maple.

Another is to use the <u>method of isoclines</u>.

An isocline is a curve of <u>constant slope</u> ie a curve (solution) of the form

$$f(x,y) = k \quad \text{for some constant } k.$$

<u>NOTE</u> Isoclines are <u>NOT</u> the same as solution curves! They are a <u>tool</u> for obtaining solution curves.

<u>Idea</u> is to draw several isoclines of the form $f(x_0, y_0) = k$ and draw little lines of slope k through each one.

Then join the lines as before.

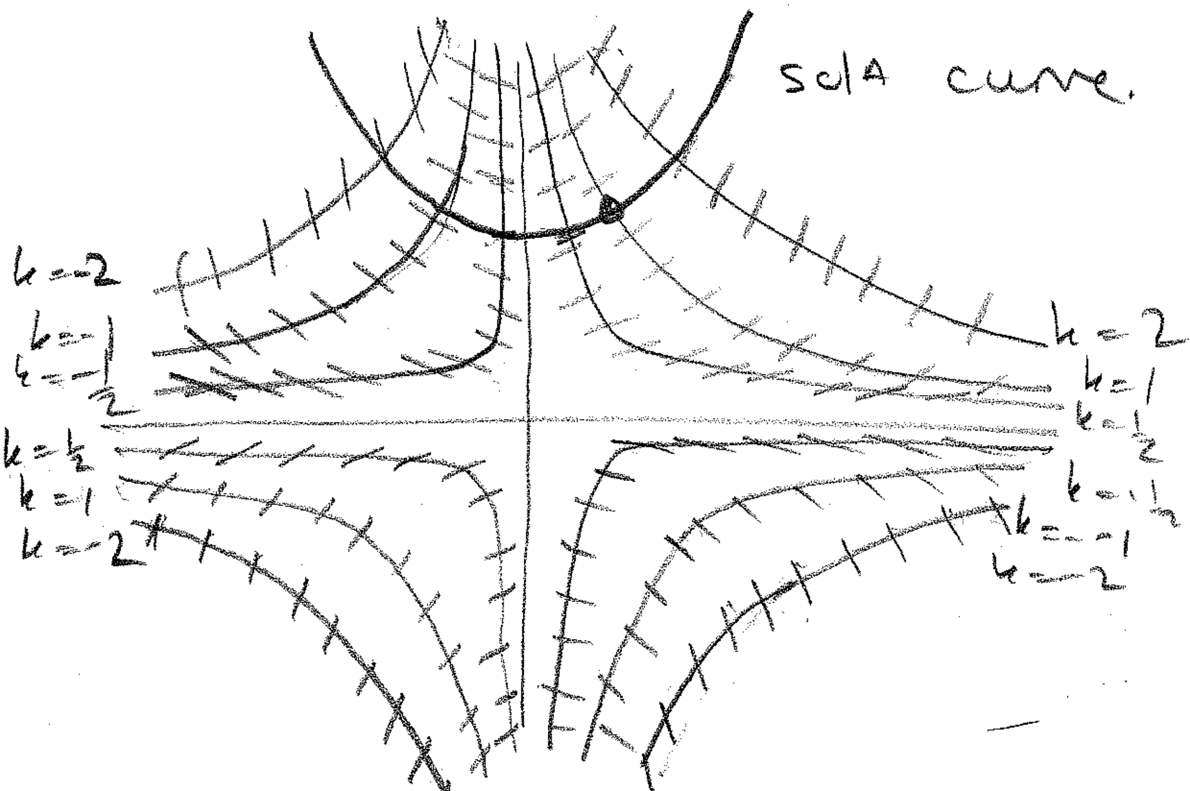
Ex. $y' = xy$.

Here the isoclines are curves of the form

$$y' = xy = k$$

ie $$y = \frac{k}{x}$$

which are hyperbolae.



Actually, we will soon be able to solve this ODE and we will find $y = ce^{x^2/2}$ where c can be fixed by adding some initial conditions

§ 1.3 Separable ODEs Modelling

Many common ODEs can be reduced to the form

$$g(y)\,y' = f(x). \qquad (1)$$

by algebraic manipulation.

We can then integrate both sides WRT x to get

$$\int g(y)\cdot y'\,dx = \int f(x)\,dx + c$$

On the lhs, if we use the subst. $y = y(x)$, so $dy = \frac{dy}{dx}\cdot dx = y'\,dx$, we get.

$$\int g(y)\,dy = \int f(x)\,dx + c.$$

Doing the integrals allows us to (implicitly) get our sol^{n} y as a f^{n} of x. This method is called <u>separating the variables</u> and an eq^{n} such as (1) is called a <u>separable eqn</u>.

Ex.

$$y' = 1 + y^2$$

This is separable as we can write

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{1}{1+y^2}\, dy = \int dx$$

$g(y) = \frac{1}{1+y^2}$ $\qquad$ $f(x) = 1$.

$$\arctan y = x + c$$

or

$$y = \tan(x+c).$$

N.b. it is important to put in the const of integration c <u>immediately after integration</u>. Putting it in later would give us instead $y = \tan x + c$ which is not a soln (if $c \neq 0$).