MTH 362-02: Fall 2004
Final Review

1. What is the polar form for $4-i$?

$$r=\sqrt{4^2+(-1)^2}=\sqrt{17} \text{ and } Arg\, z=\arctan\left(\frac{4}{-1}\right)\approx -1.32582 \text{ radians}$$

hence

$$4-i=\sqrt{17}(\cos(-1.32582)+i\sin(-1.32582))$$

2. What is the standard form of the complex number $-13.5(\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3}))$?

$$-13.5\left(\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3})\right)=-\frac{27}{2}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-\frac{27}{4}-\frac{27\sqrt{3}}{4}i$$

3. Given the following matrices

$$B=\begin{bmatrix} 4 & -12 \\ 1 & -3 \\ -3 & 9 \end{bmatrix} \quad A=\begin{bmatrix} 2 & 3 \\ -4 & -1 \end{bmatrix}$$

(a) Multiply matrices A and B, to get AB. Product does not exist, # of columns of $B=2$ is not equal to the # rows of $A=3$.

(b) Does BA exist? Justify your answer.

$$BA=\begin{bmatrix} 56 & 24 \\ 14 & 6 \\ -42 & -18 \end{bmatrix}$$

(c) Are the columns of matrix A linearly indenpendent? Justify your answer.
Yes, rank $A=2=2$

4. Compute the determinant of

$$A=\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{bmatrix}$$

$$D=\begin{vmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{vmatrix}=(1)\cdot\begin{vmatrix} 2 & 5 \\ 8 & 0 \end{vmatrix}-(0)\cdot\begin{vmatrix} 1 & 5 \\ 6 & 0 \end{vmatrix}+(2)\cdot\begin{vmatrix} 1 & 2 \\ 6 & 8 \end{vmatrix}=-48$$

5. Find the rank of the augmented matrix and discuss all solutions of the system:

$$\begin{aligned} x_1 &+ x_2 + x_3 - x_4 = 1 \\ x_1 &+ 2x_2 - 2x_3 + x_4 = 1 \\ x_1 &+ 3x_2 - 3x_3 - x_4 = 1 \\ x_1 &+ 4x_2 - 4x_3 - x_4 = 1 \end{aligned}$$

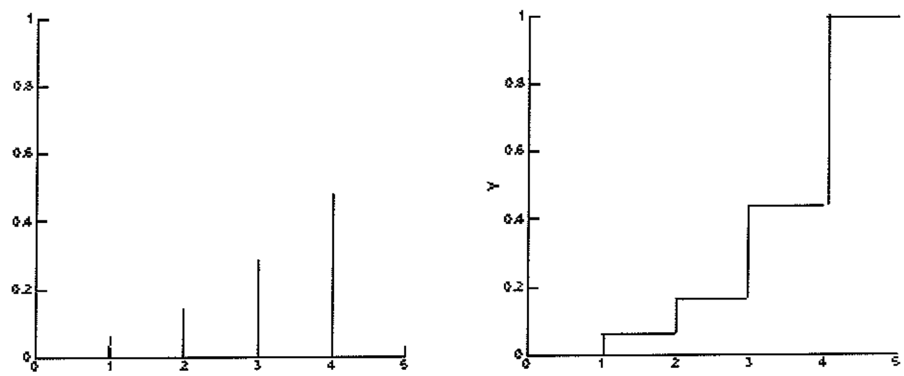
The augmented matrix $\tilde{A}$ of the system in row echelon form is

$$\tilde{A}=\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 0 \\ 0 & 0 & 0 & 8 & 0 \end{array}\right]$$

Now Rank $A =$ Rank $\tilde{A} = 4 = 4$, hence the system has a unique solution. Using back Substitution, the solution is given by

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

6. (a) Sketch the probability function $f(x) = \frac{x^2}{30}(x = 1, 2, 3, 4)$ and the distribution function
left, probability function; right, distribution function



(b) Find the mean and the variance of the random variable X for the probability function in (a).
$f(0) = 0, f(1) = \frac{1}{30}, f(2) = \frac{4}{30}, f(3) = \frac{9}{30}, f(4) = \frac{16}{30}$

$$\begin{aligned} \mu &= 0 \cdot 0 + 1 \cdot \frac{1}{30} + 2 \cdot \frac{4}{30} + 3 \cdot \frac{9}{30} + 4 \cdot \frac{16}{30} \\ &= \frac{10}{3} \\ &= 3.\overline{33} \end{aligned}$$

$$\begin{aligned} \sigma &= (1 - \frac{10}{3})^2 \cdot \frac{1}{30} + (2 - \frac{10}{3})^2 \cdot \frac{4}{30} + (3 - \frac{10}{3})^2 \cdot \frac{9}{30} + (4 - \frac{10}{3})^2 \cdot \frac{16}{30} \\ &= \frac{49}{270} + \frac{16}{270} + \frac{9}{270} + \frac{64}{270} \\ &= \frac{138}{270} = 0.001893 \end{aligned}$$

7. About 6% of the bolts produced by a certain machine are defective. Find the probabilities that in a sample of 100 bolts, the following numbers are defective.

(a) 3 or fewer. Solve first by using the binomial distribution model and then by using the Poisson distribution.

Let X be the random variable that an item is defective. The probability distribution is a binomial distribution:

$$f(x) = \binom{100}{x} (0.06)^x (0.94)^{100-x}$$

The probability of getting 3 or fewer defectives is

$$\begin{aligned} P(X \leq 3) &= \binom{100}{0}(0.06)^0(0.94)^{100} + \binom{100}{1}(0.06)^1(0.96)^{99} \\ &\quad + \binom{100}{2}(0.06)^2(0.96)^{98} + + \binom{100}{3}(0.06)^3(0.96)^{97} \\ &= 0.0020 + 0.0132 + 0.0410 + 0.0809 = 0.1371. \end{aligned}$$

This can be approximated by a poisson distribution with $\mu = np = (100)(0.06) = 6$:

$$\begin{aligned} P(x \leq 3) &= \frac{6^0}{0!}e^{-6} + \frac{6^1}{1!}e^{-6} + \frac{6^2}{2!}e^{-6} + \frac{6^3}{3!}e^{-6} \\ &= e^{-6}(1 + 6 + 18 + 36) \approx 0.15 \end{aligned}$$

(b) exactly 11

$$P(X = 11) = \binom{100}{11}(0.06)^{11}(0.94)^{89} \approx 0.0209$$

8. The amount of time between taking a pain reliever and getting relief is normally distributed with a mean of 23 minutes and a standard deviation of 4 minutes. Find the probability that the time between taking the medication and getting relief is as follows.

(a) at least 30 minutes

$P(X \geq 30) = 1 - P(X \leq 30) = 1 - \Phi\left(\frac{30-23}{4}\right) = 1 - \Phi(1.74) = 1 - 0.9599 = 0.0401,$ about 4%.

(b) at most 20 minutes

$P(X \leq 20) = \Phi(\frac{20-23}{4}) = \Phi(-0.75) = 1 - \Phi(0.75)) = 1 - 0.7734 = .2266$, about 23%.

9. Let X be normal with mean 10 and standard deviation 2. Determine c such that

(a) $P(X \leq c) = 95\%$

$$\begin{aligned} \Phi\left(\frac{c-10}{2}\right) &= 95\% \\ \frac{c-10}{2} &= 1.645 \\ c &= 13.29 \end{aligned}$$

(b) $P(X \leq c) = 5\%$

$$\begin{aligned} \Phi\left(\frac{c-10}{2}\right) &= 5\% \\ \frac{c-10}{2} &= -1.645 \\ c &= 6.71 \end{aligned}$$

(c) $P(X \leq c) = 99.5\%$

$$\begin{aligned} \Phi\left(\frac{c-10}{2}\right) &= 99.5\% \\ \frac{c-10}{2} &= 2.576 \\ c &= 15.152 \end{aligned}$$

10. Solve the separable equation and the initial value problem

(a) $y' + 3x^2y = 0$

$$\begin{aligned} \frac{dy}{dx} &= -3x^2y \\ \frac{dy}{y} &= -3x^2\,dx \\ \int \frac{dy}{y} &= \int -3x^2\,dx \\ \ln y &= -x^3 + C \\ y &= Ae^{-x^3/3} \end{aligned}$$

(b) $e^xy' = 2(x+1)y^2, \quad y(0) = \frac{1}{6}$

$$\begin{aligned} e^x\frac{dy}{dx} &= 2(x+1)y^2 \\ \frac{dy}{y^2} &= \frac{2x+2}{e^x}dx = (2xe^{-x} + 2e^{-x})\,dx \\ \int \frac{dy}{y^2} &= \int (2xe^{-x} + 2e^{-x})\,dx \\ -y^{-1} &= -2xe^{-x} - 4e^{-x} + C = \frac{1}{e^x}(-2x - 4 + Ce^x) \\ y &= \frac{e^x}{2x + 4 + Ce^x} \end{aligned}$$

Using the initial condition to solve for C

$$\begin{aligned} y(0) = \frac{1}{6} &= \frac{e^0}{2\cdot 0 + 4 + Ce^0} \\ \frac{1}{6} &= \frac{1}{C+4} \\ 6 &= C + 4 \\ C &= 2 \end{aligned}$$

so $y = \dfrac{e^x}{2x + 4 + 2e^x}$.

11. Solve the following linear differential equations and the Bernoulli equation.

(a) $y' + 2xy = 4x$

Integrating factor $e^{\int 2x dx} = e^{x^2}$. Hence

$$\begin{aligned} \frac{d}{dx}\left(ye^{x^2}\right) &= 4xe^{x^2} \\ \int \frac{d}{dx}\left(ye^{x^2}\right) &= \int 4xe^{x^2}\,dx \\ ye^{x^2/2} &= 2e^{x^2} + C \\ y &= 2 + Ce^{x^2} \end{aligned}$$

(b) $xy' = y^2 + y$

Using the sustitution $\dfrac{y}{x} = u$, we have $y = ux$ hence $y' = u + xu'$. Substituting into the ODE, we have

$$\begin{aligned} x(u + xu') &= (ux)^2 + ux \\ xu + x^2u' &= u^2x^2 + ux \\ u' &= u^2 \\ \frac{du}{dx} &= u^2 \end{aligned}$$

This is a separable equation.

$$\begin{aligned} \frac{du}{u^2} &= dx \\ \int u^{-2}du &= \int dx \\ -u^{-1} &= x + C \\ -\frac{1}{u} &= x + C \\ -\frac{x}{y} &= x + C \\ y &= -\frac{x}{x+C} \end{aligned}$$

(c) $y' + 3y = \sin x, \quad y(\frac{\pi}{2}) = 0.3$

Integrating factor $e^{\int 3dx} = e^{3x}$. Hence

$$\begin{aligned} \frac{d}{dx}\left(ye^{3x}\right) &= e^{3x}\sin x \\ \int \frac{d}{dx}\left(ye^{3x}\right) &= \int e^{3x}\sin x\, dx \\ ye^{3x} &= \frac{e^{3x}}{10}(3\sin x - \cos x) + C \\ y &= \frac{1}{10}(3\sin x - \cos x) + Ce^{-3x} \end{aligned}$$

Using the initial condition to solve for C

$$\begin{aligned} y(\frac{\pi}{2}) = 0.3 &= 0.1(3\sin\frac{\pi}{2} - \cos\frac{\pi}{2}) + Ce^{\pi/2} \\ 0.3 &= 0.3 + Ce^{\pi/2} \\ C &= 0 \end{aligned}$$

so $y = \dfrac{1}{10}(3\sin x - \cos x)$.

(d) $y' + xy = xy^{-1}$

This is a Bernoulli equation with $\alpha = -1$. Let $u = [y(x)]^{1-(-1)} = [y(x)]^2$ and the equation becomes

$$\begin{aligned} u'(1 - (-1))xu &= (1 - (-1))x \\ u' + 2xu &= 2x \end{aligned}$$

This is a first-order linear ODE. Integrating factor is $e^{\int 2x dx} = e^{x^2}$.

$$\begin{aligned} \frac{d}{dx}\left(ue^{x^2}\right) &= 2xe^{x^2} \\ \int \frac{d}{dx}\left(ue^{x^2}\right) &= \int 2xe^{x^2}\, dx \\ ue^{x^2} &= e^{x^2} + C \\ u &= 1 + Ce^{x^2} \\ y^2 &= 1 + Ce^{x^2} \end{aligned}$$

12. Show that $y_1(t) = \sqrt{t}$ and $y_2(t) = \frac{1}{t}$ are solutions of the differential equation

$$2t^2y'' + 3ty' - y = 0.$$

Show that the $y_1(t)$ and $y_2(t)$ are a linearly independent set of solutions for the ODE.

$$\begin{array}{ll} y_1'(t) = \frac{1}{2}t^{-1/2} & y_2'(t) = -\frac{1}{t^2} \\ y_1''(t) = -\frac{1}{4}t^{-3/2} & y_2''(t) = \frac{2}{t^3} \end{array}$$

Substituting y_1 and its derivatives into the ODE, we have

$$2t^2 \cdot (-\frac{1}{4})t^{-3/2} + 3t(\frac{1}{2}t^{-1/2}) - t^{1/2} = -\frac{1}{2}\sqrt{t} + \frac{3}{2}\sqrt{t} - \sqrt{t} = 0.$$

Substituting y_2 and its derivatives into the ODE, we have

$$2t^2(\frac{2}{t^3}) + 3t(-\frac{1}{t^2}) - \frac{1}{t} = \frac{4}{t} - \frac{3}{t} - \frac{1}{t} = 0.$$

To determine if the solutions are linearly independent, we solve for the Wronskian

$$W[y_1, y_2] = \begin{vmatrix} \sqrt{t} & \frac{1}{t} \\ \frac{1}{2}t^{-1/2} & -\frac{1}{t^2} \end{vmatrix} = -\frac{\sqrt{t}}{t^2} - \frac{1}{2t\sqrt{t}} = \frac{-2\sqrt{t} - \sqrt{t}}{t^2} = \frac{-3\sqrt{t}}{t^2} = -\frac{3}{t^{3/2}} \neq 0.$$

So the solutions are linearly independent.

13. Solve the following second order homogeneous equations

   (a) $y'' + 4y' - 21y = 0$
   The characteristic equation is $\lambda^2 + 4\lambda - 21 = 0$ using the quadratic formula its solution is given by $\lambda_1 = -7$ and $\lambda_2 = 3$ (case I) so the general solution is given by

$$y = C_1e^{-7x} + C_2e^{3x}$$

   (b) $y'' - 2y' + 2y = 0$ The characteristic equation is $\lambda^2 - 2\lambda + 2 = 0$ using the quadratic formula its solution is given by $\lambda = 1 \pm i$ (case III) so the general solution is given by

$$y = e^x(A\cos x + B\sin x)$$

   (c) $\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 3, y'(0) = -3 \end{cases}$

The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ using the quadratic formula its solution is given by $\lambda = -1$ (case II) so the general solution is given by

$$y = (C_1 + C_2 x)e^{-x}$$

To solve for the constants we use the two given conditions:

$$\begin{aligned} y(0) = 3 &= (C_1 + C_2 \cdot 0)e^0 = C_1 \\ C_1 &= 3 \end{aligned}$$

Now $y' = C_2 e^{-x} - (3 + C_2 x)e^{-x}$. so

$$\begin{aligned} y'(0) = -3 &= C_2 \cdot e^0 - (3 + C_2 \cdot 0)e^0 \\ C_2 - 3 &= -3 \\ C_2 &= 0 \end{aligned}$$

So the solution is $y = 3e^{-x}$.

14. (a) The initial value problem is

$$10y'' + 7y' + 100y = 0 \quad y(0) = 0.5, y'(0) = 1$$

(b) The solution to the IVP is

$$y(t) = e^{-0.35t}\left(0.5\cos(3.1428t) + 0.3739\sin(3.1428t)\right)$$

$\lim_{t\to\infty} y(t) = 0$; this limit is to be expected since damping dissipates energy causing the motion to decrease.

15. Find the general solution of the following equations using variation of parameters:

(a) $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^3}$

Solving the homogeneous equation $y'' + 4y' + 4y = 0$, the linearly independent solutions are $y_1 = e^{-2x}, y_2 = xe^{-2x}$ so $W(y_1, y_2) = e^{-4x}$ Then

$$\begin{aligned} \frac{y_2 r}{W} &= \frac{e^{-2x} x e^{-2x}}{x^3 e^{-4x}} = \frac{1}{x^2} \quad \Rightarrow \quad u = \int \frac{1}{x^2} = -x^{-1} \\ \frac{y_1 r}{W} &= \frac{e^{-2x} e^{-2x}}{x^3 e^{-4x}} = \frac{1}{x^3} \quad \Rightarrow \quad v = \int \frac{1}{x^3} = -\frac{1}{2x^2} \end{aligned}$$

Thus

$$y_p(x) = -y_1 \cdot u + y_2 \cdot v = \frac{e^{-2x}}{x} + \frac{xe^{-2x}}{2x^2} = \frac{e^{-2x}}{2x}$$

and the general solution is

$$y(x) = e^{-2x}\left[c_1 - c_2 x + \frac{1}{2x}\right]$$

(b) $y'' + y = \tan x, \quad 0 < x < \dfrac{\pi}{2}$

The functions $y_1 = \cos x$ and $y_2 = \sin x$ are two linearly independent solutions of the homogeneous equation $y'' + y = 0$ with

$$W(y_1, y_2) = (\cos x)\cos x - (-\sin x)\sin x = 1$$

$$\frac{y_2 r}{W} = \tan x \sin x \quad \Rightarrow \quad u = \int \tan x \sin x = \ln(\sec x + \tan x) - \sin x$$
$$\frac{y_1 r}{W} = \tan x \cos x \quad \Rightarrow \quad v = \int \tan x \cos x = -\cos x$$

Thus a particular is

$$y_p(x) = -\cos x \ln(\sec x + \tan x)$$

and the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln(\sec x + \tan x).$$

16. Find a particular solution of the following equation using the method of undetermined coefficients:

(a) $y'' + y' + y = x^2$

First we solve for the solutions to the homogeneous equations $y''+y'+y = 0$. Its characteristic equation is $\lambda^2 + \lambda + 1 = 0$ whose solutions is given by $\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ so

$$y_h = e^{-\frac{1}{2}x}(C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x)$$

Since $r(x) = x^2$ does not appear in $y_h(x)$, we try a particular solution of the form $y_p = k_2x^2 + k_1x + k_0$. Differentiating twice we have

$$\begin{aligned} y_p' &= 2k_2x + k_1 \\ y_p'' &= 2k_2 \end{aligned}$$

Substituting into the ODE and rearranging terms we get

$$2k_2 + 2k_2x + k_1 + k_2x^2 + k_1x + k_0 = k_2x^2 + (2k_2 + k_1)x + (2k_2 + k_1 + k_0) = x^2$$

Comparing coefficients, we get

$$\begin{aligned} k_2 &= 1 \\ 2k_2 + k_1 &= 0 \\ 2k_2 + k_1 + k_0 &= 0 \end{aligned}$$

The solution to the system is

$$k_2 = 1, k_1 = -2, k_0 = 0$$

So a particular solution is $y_p(x) = x^2 - 2x$

(b) $3y'' + 10y' + 3y = 9x + 5\cos x$

First we solve for the solutions to the homogeneous equations $3y'' + 10y' + 3y = 0$. Its characteristic equation is $3\lambda^2 + 10\lambda + 3 = 0$ whose solutions is given by $\lambda_1 = -\frac{1}{3}$ and $\lambda_2 = -3$ hence

$$y_h = e^{-\frac{1}{3}x} + e^{-3x}$$

Since $r(x)$ does not appear in $y_h(x)$, we try a particular solution of the form $y_p = k_1x + k_0 + M_1 \cos x + M_2 \sin x$. Differentiating twice we have

$$\begin{aligned} y_p' &= k_1 - M_1 \sin x + M_2 \cos x \\ y_p'' &= -M_1 \cos x - M_2 \sin x \end{aligned}$$

Substituting into the ODE and rearranging terms we get

$$3k_1x + (10k_1 + 3k_0) + 10M_2\cos x - 10M_1\sin x = 9x + 5\cos x$$

Comparing coefficients, we get

$$\begin{aligned} M_2 &= \frac{1}{2} \\ M_1 &= 0 \\ k_1 &= 3 \\ k_0 &= -\frac{10}{9} \end{aligned}$$

So a particular solution is $y_p(x) = 3x - \frac{10}{9} + \frac{1}{2}\sin x$