Chapter 4

Systems of ODEs

First, some more linear algebra which we will need later.

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix (with real or complex entries), let λ be a scalar (real or complex) and consider the matrix eq^n.

$$A\underline{x} = \lambda \underline{x}.$$

If this eq^n has a sol^n $\underline{x}$ which is not the zero vector $\underline{0}$, we say λ is an <u>eigenvalue</u> of A and x is an <u>eigenvector</u> for this eigenvalue.

Finding Eigenvalues and Eigenvectors

Back to

$$A\underline{x} = \lambda \underline{x}.$$

Can rewrite this slightly as

$$A\underline{x} = (\lambda I_n)\underline{x}$$

where $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ is the $n \times n$ identity matrix.

Now subtract $(\lambda I_n)\underline{x}$ from both sides

$$A\underline{x} - (\lambda I_n)\underline{x} = \underline{0}$$

$$(A - \lambda I_n)\underline{x} = \underline{0}.$$

Now

$(A - \lambda I_n)\underline{x} = \underline{0}$ has a sol^n for some $\underline{x} \neq \underline{0}$

$$\Updownarrow$$

$\text{Rank}(A - \lambda I_n) < n$ (§7.5 Thm 2, p.304)

$$\Updownarrow$$

$\det(A - \lambda I_n) = 0$ (§7.7 Thm 3, p. 311)

Thus λ is an eigenvalue (e-value) of A iff it is a sol^n of the eq^n.

$$\det(A - \lambda I_n) = 0.$$

This is a polynomial of degree n known as the <u>characteristic equation</u> of the matrix A.

The procedure for finding eigenvalues (e-values) and eigenvectors (evectors) is then as follows.

How to Find all Eigenvalues and Eigenvectors of a Matrix

Step 1 Calculate the characteristic eq$^{\underline{n}}$.

$$\det(A - \lambda I) = 0$$

and find all its roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ $(m \leq n)$. These are the e-values of A.

Step 2 For each fixed eigenvalue λ_i, $1 \leq i \leq m$, solve the homogeneous linear system

$$(A - \lambda_i I)\underline{x} = 0$$

(e.g. row reduction, reduced echelon form). and find a basis for the sol$^{\underline{n}}$ space. This gives the e-vectors.

Ex. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

Soln The char eqn is the quadratic eqn.

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -4.0-\lambda & 4.0 \\ -1.6 & 1.2-\lambda \end{vmatrix} = 0$$

$$(-4.0-\lambda)(1.2-\lambda) - 4.0(-1.6) = 0$$

$$\lambda^2 + 2.8\lambda + 1.6 = 0.$$

This has roots $\lambda_1 = -2$ and $\lambda_2 = -0.8$ and so these are the e-values of A.

To find the e-vector for $\lambda_1 = -2$, we have

$$\begin{bmatrix} -4.0-(-2) & 4.0 \\ -1.6 & 1.2-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2.0 & 4.0 \\ -1.6 & 3.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2.0\, x_1 + 4.0 x_2 = 0$$
$$-1.6\, x_1 + 3.2\, x_2 = 0$$

This clearly gives a single eqn.

$$-x_1 + 2x_2 = 0$$

for which a soln is $x_1 = 2$, $x_2 = 1$

and so

$$\underline{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is an e-vector.

To find the e-vector for the other e-value $\lambda_2 = -0.8$, we have

$$\begin{bmatrix} -4.0-(-0.8) & 4.0 \\ -1.6 & 1.2-(-0.8) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3.2 & 4.0 \\ -1.6 & 2.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3.2x_1 + 4.0x_2 = 0$$
$$-1.6x_1 + 2.0x_2 = 0$$

Again we get one eqΔ

$$-4x_1 + 5x_2 = 0$$

for which a solΔ is $x_1 = 5, x_2 = 4$ and so

$$\underline{x}^{(2)} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

is an e-vector.

To summarize

$$A = \begin{bmatrix} -4.0 & 4.0 \\ -1.6 & 1.2 \end{bmatrix}$$

Has two eigenvectors $\lambda_1 = -2.0$, $\lambda_2 = -0.8$
with associated eigenvectors

$$\underline{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \underline{x}^{2} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad \text{(respectively)}.$$

EXAMPLE: Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Step 1. Find the eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} = (2-\lambda)^2(1-\lambda) = 0.$$

Eigenvalues of A: $\lambda = 1$ and $\lambda = 2$.

Step 2. Find three linearly independent eigenvectors of A.

By solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$, for each value of λ, we obtain the following:

Basis for $\lambda = 1$: $\quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Basis for $\lambda = 2$: $\quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Step 3: Construct P from the vectors in step 2.

$$P = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Step 4: Construct D from the corresponding eigenvalues.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Step 5: Check your work by verifying that AP = PD

$$\boldsymbol{AP} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\boldsymbol{PD} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$