Chapter 3

Higher Order Linear ODEs

We extend the methods we developed in Chapter 2 for second order linear ODEs to third and higher order ODEs.

§ 3.1 Homogeneous Linear ODEs

An ODE is nth order if the highest order derivative of the unknown fn y which appears is the nth derivative $y^n = \frac{d^n y}{dx^n}$. Such an ODE can be written

$$F(x, y, y', \ldots, y^{(n)}) = 0 \qquad \left(y^{(i)} = \frac{d^i y}{dx^i} \right).$$

Such an ODE is linear if it can be written

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = r(x).$$

The fns $p_{n-1}(x), \ldots, p_0(x)$ are called the coefficients of the ODE. Note that in this form the ODE is a linear fn of $y, y', \ldots, y^{(n)}$ each taken individually. (ie hold the others fixed).

If $r(x) \equiv 0$, the eqn becomes

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = 0.$$

In this case we say the ODE is homogeneous. Otherwise ($r(x) \not\equiv 0$), the ODE is inhomogeneous.

A <u>solution</u> of an nth order ODE

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

on an open interval I is a fn $h(x)$ which is defined and n times differentiable on I s.t. if we replace y by $h(x)$, $y'(x)$ by $h'(x)$ and so on, then

$$F(x, h(x), h'(x), \ldots, h^{(n)}(x)) = 0$$

everywhere on I.

Homogeneous Linear ODEs

Superposition Principle, General Solution

Here we consider the nth order homog. linear ODE

$$\circledast \quad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$

Thm 1 (Superposition Principle).

Any linear combination of sol^ns of $\circledast$ on some open interval I is again a sol^n on I. (In other words, the sol^ns of $\circledast$ on I form a vector space).

Pf. Similar to the second order case (Thm 1, P. 47).

$\underline{\text{Defn}}$. Linear Independence

n fns $y_1(x), \ldots, y_n(x)$ are called <u>linearly independent</u> on some interval I if the eqn.

$$k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) \equiv 0 \quad \text{on } I$$

implies that $k_1 = k_2 = \cdots = k_n = 0$.

Otherwise we say $y_1(x), \ldots, y_n(x)$ are <u>linearly dependent</u> on I and in this case we can find consts $k_1, \ldots, k_n$ not all 0 s.t.

$$k_1 y_1(x) + k_2 y_2(x) + \cdots + k_n y_n(x) \equiv 0 \quad \text{on } I.$$

<u>Def</u>

Defn. A <u>basis</u> or <u>fundamental system</u> of solns of the ODE (*) on an open interval I is a set of n solutions $y_1(x), y_2(x), \ldots, y_n(x)$ of (*) which is lin ind. on I.

A <u>general soln</u> of (*) on I is a soln of the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

where $y_1, \ldots, y_n$ is a basis of solns on I and the consts $C_1, C_2, \ldots, C_n$ are arbitrary.

A <u>particular soln</u> of (*) on I is obtained if we assign values to the constants $C_1, \ldots, C_n$.

Initial Value Problem. Existence and Uniqueness

An initial value problem (IVP) for the ODE ⊛ consists of ⊛ and n initial conditions

$$(**) \quad y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x) = 0$$

$$y(x_0) = K_0,\ y'(x_0) = K_1, \ldots, y^{(n-1)}(x_0) = K_{n-1}$$

where x_0 & $K_0, \ldots, K_{n-1}$ are given.

n.b. If the ODE is nth order, then the IC's go up to derivatives of order $n-1$.

Thm 2 Existence and Uniqueness Thm for IVPs

If the coeffs $p_0(x), \ldots, p_{n-1}(x)$ in $(**)$ are cts on some open interval I and $x_0 \in I$, then the IVP $(**)$ has a unique sol^{n} $y(x)$ on I.

Linear Independence of Solutions:

Wronskian

As in chapter 2, linear independence of solns of an nth order ODE is determined by the behaviour of their Wronskian.

For n fcs $y_1(x), \ldots, y_n(x)$ which are defined and n-1 times diff on an open interval I, their Wronskian $W = W(y_1, \ldots, y_n)$ is given by the nth order determinant

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

n.b. This generalises the defn of $W(y_1, y_2)$ for $n=2$ which we had in Chapter 2.

<u>Thm 3</u> Linear Dependence and Independence of Sol^{n}s

Suppose the coeffs. $p_0(x), \ldots, p_{n-1}(x)$ of the ODE ⊛ are cts. on some open interval I. Then n sol^{n}s $y_1, \ldots, y_n$ of ⊛ on I are lin dep. iff their Wronskian is 0 for some $x_0 \in I$.

Furthermore, if W is 0 at some pt $x = x_0$ then $W \equiv 0$ on I. Hence if $\exists\, x_1 \in I$ at which $W \neq 0$, then $y_1, \ldots, y_n$ are lin ind on I and form a basis of sol^{n}s of ⊛ on I.

Pf. Similar to that of Thm 2, P. 74.

Ex. For the 4th order ODE

$$y^{iv} - 5y'' + 4y = 0,$$

we will see (soon) that

$$e^{-2x}, e^{-x}, e^{x}, e^{2x}$$

are all sol$^{\text{ns}}$. Their Wronskian is

$$W = \begin{vmatrix} e^{-2x} & e^{-x} & e^{x} & e^{2x} \\ -2e^{-2x} & -e^{-x} & e^{x} & 2e^{2x} \\ 4e^{-2x} & e^{-x} & e^{x} & 4e^{2x} \\ -8e^{-2x} & -e^{-x} & e^{x} & 8e^{2x} \end{vmatrix}$$

$$= e^{-2x} e^{-x} e^{x} e^{2x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix}$$

$$= 1 \cdot 72$$

$$= 72.$$

Thus these solns are lin ind (on $\mathbb{R}$) and the general soln of the ODE is

$$y = c_1 e^{-2x} + c_2 e^{-x} + c_3 e^{x} + c_4 e^{2x}.$$

A General Soln of ⊛ Includes all Solns

Thm 4 Existence of a General Soln

If the coeffs $p_0(x), \ldots, p_{n-1}(x)$ of ⊛ are cts on some open interval I, then ⊛ has a general soln on I.

Pf. Similar to that of Thm 3 on p. 76.

Thm 5 A General Soln Includes all Solns.

If ⊛ has cts coeffs $p_0(x), \ldots, p_{n-1}(x)$ on some open interval I, then every soln $y = Y(x)$ of ⊛ is of the form

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$$

where $y_1, \ldots, y_n$ is a basis of sol^{n}s of $\circledast$ on I and $C_1, \ldots, C_n$ are suitable consts.