§ 2.2 Homogeneous Linear ODEs with Constant Coefficients

We consider 2nd order homog. linear ODEs of the form

$$y'' + ay' + by = 0$$

where the coefficients a, b are consts.

Recall that for the first order version

$$y' + ky = 0,$$

the solution was an exponential fn.

$$y = e^{-kx}.$$

This suggests we try using an exponential fn as a sol^n for the second order case as well.

So let $y = e^{\lambda x}$. Then

$$y' = \lambda e^{\lambda x}, \qquad y'' = \lambda^2 e^{\lambda x}$$

and if we subst. this into the ODE we get

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$$

Factoring gives

$$(\lambda^2 + a\lambda + b) \cdot e^{\lambda x} = 0$$

and since $e^{\lambda x} \neq 0$ (as exponential fns can never be 0), we must have

$$\lambda^2 + a\lambda + b = 0$$

In other words, we have reduced solving the ODE to solving a quadratic eqn, known as the characteristic eqn

By the formula for the roots of a quadratic eq^n, we have two roots

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2-4b}\right)$$

$$\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2-4b}\right)$$

and $e^{\lambda_1 x}$, $e^{\lambda_2 x}$ are both sol^{ns} of the ODE (which you can check directly).

There are 3 basic cases to consider which are governed by the <u>discriminant</u> a^2-4b

<u>Case I</u> $a^2-4b>0$ - two real roots

<u>Case II</u> $a^2-4b=0$ - one real double root

<u>Case III</u> $a^2-4b<0$ - complex conjugate roots.

Case I Two Distinct Real Roots λ_1, λ_2

Here we have two solutions

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$$

and since $\lambda_1 \neq \lambda_2$, the quotient

$$\frac{y_2}{y_1} = e^{(\lambda_2 - \lambda_1)x}$$ is not a constant fn

and so y_1, y_2 are lin. ind.

Thus the general sol^n of the ODE in this case is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

Ex. Solve the IVP

$$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5$$

Step 1. General solⁿ

The char eqⁿ is

$$\lambda^2 + \lambda - 2 = 0$$

which factorizes as

$$(\lambda - 1)(\lambda + 2) = 0.$$

Thus $\lambda_1 = 1, \quad \lambda_2 = -2$

and the general solⁿ is

$$y = c_1 e^{x} + c_2 e^{-2x}.$$

Step 2 **Particular Sol^{n}**

Need
$$y' = c_1 e^{x} - 2c_2 e^{-2x}.$$

$$y(0) = 4 \Rightarrow c_1 + c_2 = 4$$
$$y''(0) = -5 \Rightarrow c_1 - 2c_2 = -5$$

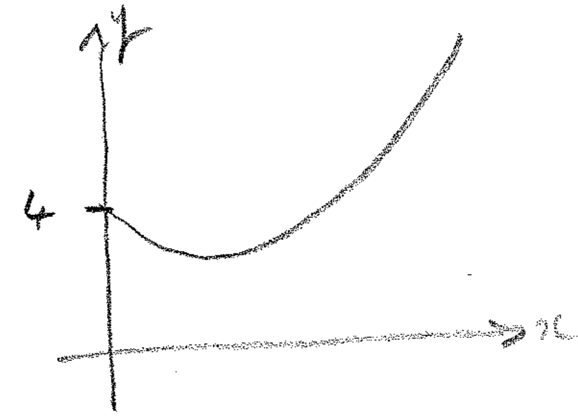
Subtract
$$3c_2 = 9$$
$$c_2 = 3$$

Subst.
$$c_1 + 3 = 4$$
$$c_1 = 1.$$

Thus
$$y = e^{x} + 3e^{-2x}$$

is the desired sol^{n}.

Soln looks like



Case II Real Double Root $\lambda = -\frac{a}{2}$

If the discriminant $a^2 - 4b$ is zero, then

$$\lambda = \lambda_1 = \lambda_2 = -\frac{a}{2}$$

and we only get one sol$^{\underline{n}}$ $y_1 = e^{-(a/2)x}$.

To look for a second lin ind. sol$^{\underline{n}}$, we use reduction of order.

So set $y_2 = u y_1$

$$y_2' = u'y_1 + u y_1', \quad y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

Subst. into the ODE

$$(u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + buy_1 = 0$$

Gather terms in u'', u', u.

$$u'' y_1 + u'(2y_1' + ay_1) + u(y_1'' + ay_1' + by_1) = 0$$

(As before) $y_1'' + ay_1' + by_1 = 0$ as y_1 is a solⁿ. In addition, since

$$y_1 = e^{-a/2\,x}, \quad y_1' = -\frac{a}{2} e^{-a/2\,x}$$

and so $$2y_1' + ay_1$$
$$= (-a + a) e^{-a/2\,x}$$
$$= 0.$$

Thus we are left with

$$u'' y_1 = 0$$

and since $y_1 = e^{-a/2\,x}$ is never 0,

$$u'' = 0.$$

Two (very easy) integrations then give

$$u = c_1 x + c_2 .$$

To get a second lin ind soln set $c_1 = 1$, $c_2 = 0$, so $u = x$ and

$$y_2 = u y_1 = x e^{-a/2\, x}.$$

Since $\frac{y_2}{y_1} = x$ is not const, y_1, y_2 are lin ind and the general soln in this case is

$$y = c_1 y_1 + c_2 y_2 = (c_1 + c_2 x) e^{-\frac{a}{2} x}.$$

<u>WARNING</u> If λ is <u>not</u> a double root (ie a simple root), then

$(c_1 + c_2 x) e^{\lambda x}$ with $c_2 \neq 0$ is <u>not</u> a soln.

Ex Solve the IVP

$$y'' + y' + \frac{y}{4} = 0 \quad , \quad y(0) = 3, \quad y'(0) = -7/2.$$

Step 1: General soln.

The char eqn is

$$\lambda^2 + \lambda + \frac{1}{4} = 0$$

$$(\lambda + \tfrac{1}{2})^2 = 0$$

which has a double root $\lambda = -\frac{1}{2}$.

The general soln is then

$$y = (c_1 + c_2 x) e^{-\frac{x}{2}}.$$

Step 2 Particular soln.

Need $y' = c_2 e^{-\frac{x}{2}} - \frac{1}{2}(c_1 + c_2 x) e^{-\frac{x}{2}}$

$$= \left(-\frac{c_1}{2} + c_2 - \frac{c_2 x}{2}\right) e^{-\frac{x}{2}}.$$

$y(0) = 3 \Rightarrow \quad c_1 = 3$

$y'(0) = -\frac{7}{2} \Rightarrow \quad -\frac{c_1}{2} + c_2 = -\frac{7}{2}$

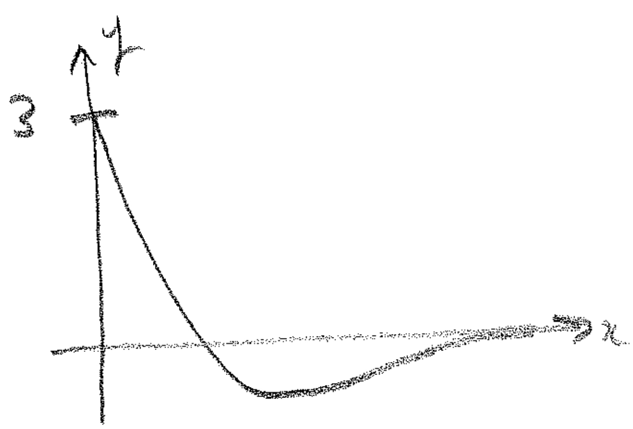
$$c_2 = -\frac{7}{2} + \frac{3}{2}$$

$$= -2$$

So the desired soln is

$$y = (3 - 2x) e^{-x/2}.$$

Soln looks like



Critical Damping

Case III Complex Roots

Here the discriminant $a^2 - 4b < 0$ and if we let ω be a +ve real no. s.t. $4\omega^2 = 4b - a^2 > 0$ (i.e. $\omega = \sqrt{b - a^2/4}$), then

$$\sqrt{a^2 - 4b} = \sqrt{-4\omega^2} = \pm 2i\omega$$

(remember that the complex $\sqrt{\ }$ has two branches)

Then

$$\lambda_1 = -\frac{a}{2} + \frac{1}{2}\sqrt{-4\omega^2} = -\frac{a}{2} + i\omega$$

$$\lambda_2 = -\frac{a}{2} - \frac{1}{2}\sqrt{-4\omega^2} = -\frac{a}{2} - i\omega$$

Remembering that $e^{x+iy} = e^x(\cos y + i\sin y)$, we get two solutions

$$\tilde{y}_1 = e^{\lambda_1 x} = e^{-\frac{a}{2}x}(\cos(\omega x) + i\sin(\omega x))$$

$$\tilde{y}_2 = e^{\lambda_2 x} = e^{-a/2\, x}(\cos(-\omega x) + i\sin(-\omega x))$$

$$= e^{-a/2\, x}(\cos(\omega x) - i\sin(\omega x)).$$

Naturally, we would prefer real-valued sol$^{\text{ns}}$. We get these by setting

$$y_1 = \frac{\tilde{y}_1 + \tilde{y}_2}{2} = e^{-a/2\, x} \cos(\omega x)$$

$$y_2 = \frac{\tilde{y}_1 - \tilde{y}_2}{2i} = e^{-a/2\, x} \sin(\omega x).$$

Since $\frac{y_2}{y_1} = \tan \omega x$ is non-const,

y_1, y_2 are lin ind. and the general sol$^{\text{n}}$ to the ODE is

$$y = c_1 e^{-\frac{a}{2}x} \cos(\omega x) + c_2 e^{-\frac{a}{2}x} \sin(\omega x)$$

or

$$y = e^{-a/2\, x} (c_1 \cos(\omega x) + c_2 \sin(\omega x))$$

$$\omega = \sqrt{b - a^2/4} \quad (>0).$$

Ex. Solve the IVP

$$y'' + y' + \tfrac{17}{4}y = 0, \quad y(0)=2, \quad y'(0)=-1$$

Step 1 General Sol$^{\underline{n}}$

Char eq^n is

$$\lambda^2 + \lambda + \tfrac{17}{4} = 0$$

which has roots

$$\lambda = -\tfrac{1}{2} \pm \sqrt{1-17}$$

$$= -\tfrac{1}{2} \pm \sqrt{-16}$$

$$= -\tfrac{1}{2} \pm 2i$$

~~Here~~ $\omega = 2$ and so the general sol$^{\underline{n}}$ is

$$y = e^{-\frac{x}{2}}\left(c_1 \cos(2x) + c_2 \sin(2x)\right).$$

Step 2 Particular Soln.

Need

$$y' = -\frac{1}{2} e^{-\frac{x}{2}} (c_1 \cos(2x) + c_2 \sin(2x))$$

$$+ e^{-x/2} (-2c_1 \sin(2x) + 2c_2 \cos(2x))$$

$$= e^{-\frac{x}{2}} \Big((-\tfrac{1}{2} c_1 + 2c_2) \cos 2x$$

$$+ (-2c_1 - \tfrac{1}{2} c_2) \sin 2x \Big)$$

$y(0) = 2$

$\Rightarrow \quad c_1 = 2$

remember

$\left(\cos(0) = 1 \quad \sin(0) = 0 \right)$

$y'(0) = -1$

$$\Rightarrow \quad -\frac{1}{2} c_1 + 2 c_2 = 0$$

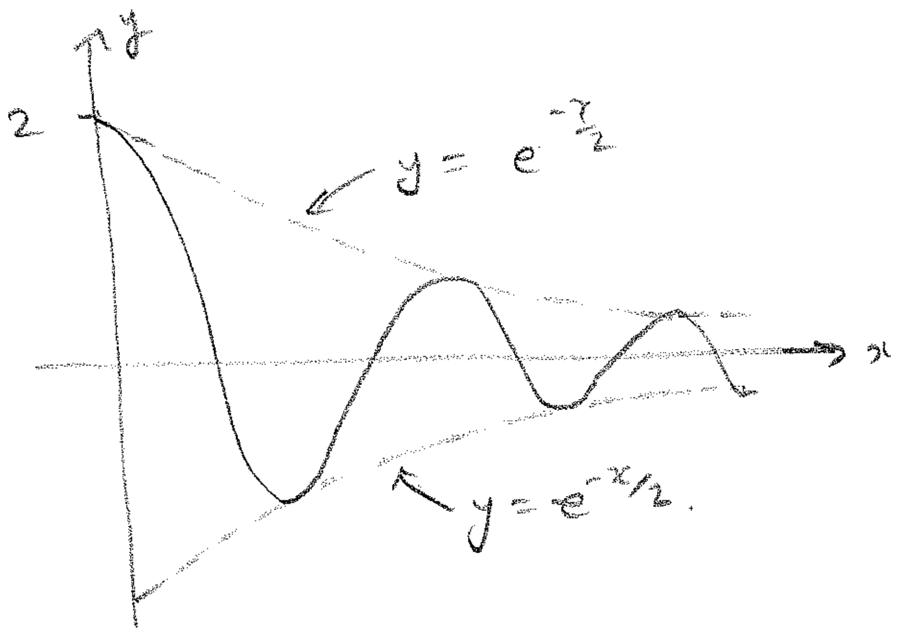
$$-1 + 2c_2 = 0$$

$$c_2 = \frac{1}{2}.$$

Hence the desired sol^n is

$$y = e^{-x/2}\left(2\cos(2x) + \frac{1}{2}\sin(2x)\right)$$

Sol^n looks like



Exponentially damped oscillations which lie in the 'envelope' between $-e^{-x/2}$ and $e^{-x/2}$.

Summary of Cases I, II, III

Case	Roots	Basis	General Soln.
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{a}{2}$	$e^{-ax/2}, x e^{-ax/2}$	$y = (c_1 + c_2 x) e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{a}{2} + i\omega$, $\lambda_2 = -\frac{a}{2} - i\omega$, $\omega = \sqrt{b - a^2/4}$	$e^{-ax/2} \cos \omega x$, $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(c_1 \cos \omega x + c_2 \sin \omega x)$