§ 2.10 Variation of Parameters

For the inhomog. 2nd order linear ODE

(1) $$y'' + p(x)y' + q(x)y = r(x)$$

where p, q, r are cts on some open interval I, a particular soln y_p on I is given by

(2) $$y_p(x) = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx .$$

Here y_1, y_2 are a basis of solns of the associated homog ODE

$$y'' + p(x)y' + q(x)y = 0$$

on I and $W = W(y_1, y_2)$ is their Wronskian.

Ex. Solve the inhomog ODE

$$y'' + y = \sec x = \frac{1}{\cos x}$$

Soln. Note first that the rhs is not one of the types we discussed when doing undetermined coeffs. (§ 2.7).

A basis of solns of the associated homog ODE on $\mathbb{R}$ is $y_1 = \cos x$, $y_2 = \sin x$

Then

$$W = W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$
$$= \cos x \cdot \cos x - \sin x \cdot -\sin x$$
$$= 1$$

From (2), choosing 0 consts. of integration, we get a particular soln.

$$y_p = -\cos x \int \frac{\sin x \sec x}{1}\, dx + \sin x \int \frac{\cos x \sec x}{1}\, dx$$

$$= -\cos x \int \frac{\sin x}{\cos x}\, dx + \sin x \int dx$$

$u = \cos x$
$du = -\sin x$

$$= -\cos x \int \frac{-du}{u} + \sin x \int dx$$

$$= \cos x \int \frac{du}{u} + \sin x \int dx$$

$$= \cos x \ln|u| + \sin x \cdot x$$

$$= \cos x \ln|\cos x| + x \sin x.$$

The general soln is then

$$y = y_h + y_p = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x.$$