13.7 The Complex Logarithm, General Powers of z

Let $z \in \mathbb{C}$ be fixed and suppose we try to solve for w the equation

$$e^{w} = z. \quad (\text{note } z \neq 0 \text{ is not allowed here!})$$

Write z in polar form as $re^{i\theta}$ and w in cartesian form as $u + iv$ so that

$$e^{u+iv} = re^{i\theta}$$

From the last section, we have that

$$|e^{w}| = |e^{u+iv}| = e^{u} = r.$$

And if we take ln of both sides of $e^{u} = r$ using the standard $\ln x$ of calculus, then

$$u = \ln r = \ln|z|.$$

Also from the last section

$$v = \arg(e^{u+iv}) = \arg(re^{i\theta}) = \theta$$

(up to integer multiples of 2π).

So

$$v = \arg(z)$$

and we <u>define</u> the complex logarithm of z, $\ln z$ $(z \neq 0)$ by

$$\ln z := \ln|z| + i \arg z.$$

<u>NOTE</u> $\ln z$ is only defined up to integer multiples of $2\pi i$ as $\arg z$ is only defined up to integer multiples.

One way around this is to use the principal branch $\operatorname{Arg} z$ of the argument $(-\pi < \theta \leq \pi)$.

This gives us the principal branch $\mathrm{Ln}\, z$ of the logarithm defined by

$$\mathrm{Ln}\, z := \ln|z| + i\,\mathrm{Arg}\, z \quad , \qquad z = re^{i\theta}$$
$$-\pi < \theta \leq \pi .$$

Remarks,

If $z = x > 0$, then

$$\mathrm{Ln}\, z = \ln x + i\,\mathrm{Arg}(x + 0i)$$
$$= \ln x,$$

so the principal branch of the logarithm extends the logarithm of calculus.

If $z = x < 0$, then

$$\ln z = \ln|x| + i\,\mathrm{Arg}(x + i0)$$
$$= \ln|x| + i\pi .$$

Note. $\ln x$ from calculus is undefined for $x < 0$.

Ex. $\ln 1 = 0 + 2n\pi i, \quad n \in \mathbb{Z}.$

$\mathrm{Ln}\, 1 = 0.$

$\ln(-1) = \pi i + 2n\pi i \quad , \; n \in \mathbb{Z}$

$= (2n+1)\pi i \quad , \; n \in \mathbb{Z}$

$\mathrm{Ln}\,(-1) = \pi i$

$\ln(4i) = \ln 4 + i\frac{\pi}{2} + 2n\pi i \quad , \; n \in \mathbb{Z}$

$= \ln 4 + i(2n + \frac{1}{2})\pi \quad , \; n \in \mathbb{Z}.$

$\mathrm{Ln}(4i) = \ln 4 + i\frac{\pi}{2}.$

Note that

$$e^{\ln z} = e^{\ln|z| + i\arg z} = |z| e^{i\arg z} = z, \quad z \neq 0$$

while

$$\ln(e^z) = \ln(e^{\mathrm{Re}z + i\mathrm{Im}z}), \quad z \in \mathbb{C}$$

$$= \ln(e^{\mathrm{Re}z} \cdot e^{i\mathrm{Im}z}), \quad z \in \mathbb{C}$$

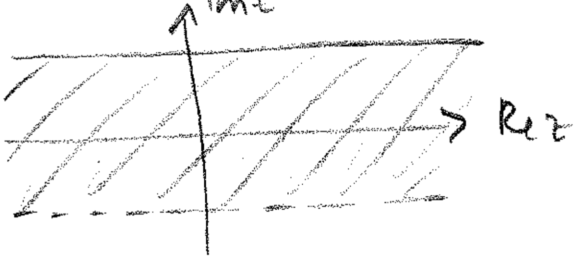
$$= \mathrm{Re}z + i\mathrm{Im}z + 2n\pi i, \quad n \in \mathbb{Z}, \; z \in \mathbb{C}.$$

Also

$$\mathrm{Ln}(e^z) = \mathrm{Ln}(e^{\mathrm{Re}z} \cdot e^{i\mathrm{Im}z})$$

$$= \mathrm{Re}z + i\mathrm{Im}z$$

$$= z, \quad \text{provided } -\pi < \mathrm{Im}z \leq \pi.$$



Shows how the logarithm and the exponential are 'inverses' of each other.

The formulae from calculus

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2, \quad \ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$$

hold, but <u>only up to integer multiples of $2\pi i$</u>.

(Can get around this partly using $\operatorname{Ln} z$, but need to be careful).

<u>General Powers of z</u>

For $c \in \mathbb{C}$ fixed, define z^c $(z \neq 0)$ by

$$z^c := e^{c \ln z}.$$

Note that this fn is not, in general, single-valued. Using $\operatorname{Ln} z$, we can define the principal branch of z^c by

$$z^c = e^{c \operatorname{Ln} z}.$$

If $c = n \in \mathbb{Z}$, then

$$z^c = e^{n \ln z} = e^{n(\ln|z| + i \arg z)}$$

$$= e^{n \ln|z|} e^{i n \arg z}$$

$$= e^{\ln(|z|^n)} e^{i n \arg z}$$

as $a \ln b = \ln(b^a)$ from calculus

$$= e^{\ln(|z|^n)} e^{i \arg z^n}$$

as $\arg z^n = n \arg z$ up to integer multiples of $2\pi i$

$$= e^{\ln|z^n|} e^{i \arg z^n} \quad \text{as } |z^n| = |z|^n$$

$$= |z|^n e^{i \arg z^n}$$

$$= z^n \quad \text{(polar form of } z^n\text{)}.$$

Hence our new definition does indeed extend 'ordinary' powers of z.

Ex.

$$i^i = e^{i \ln i} = e^{i\left(\frac{\pi}{2} + 2n\pi i\right)}, \quad n \in \mathbb{Z}$$

$$= e^{-\left(\frac{\pi}{2} + 2n\pi\right)}, \quad n \in \mathbb{Z}.$$

and the principal value (branch) is

$$e^{-\frac{\pi}{2}}.$$

Lastly, note that in a similar way we can define the fn a^z, $z \neq 0$ by

$$a^z := e^{z \ln a},$$