Also

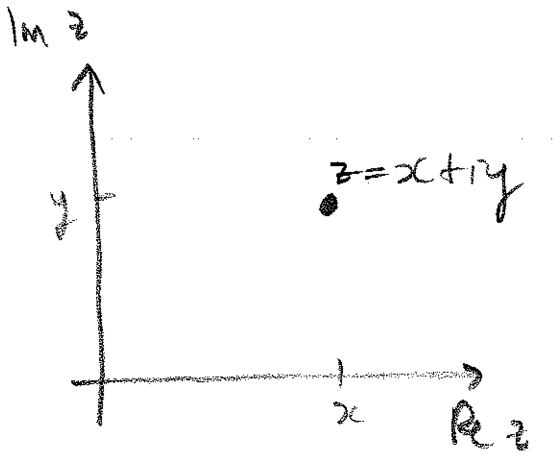
$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

which gives us another way of deriving the formula for dividing complex nos.

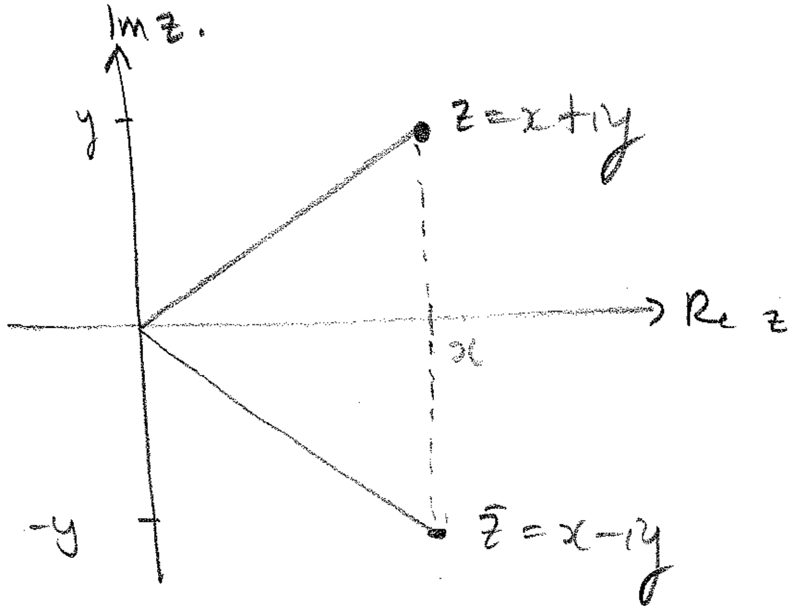
13.2 The Complex Plane, Polar Form, Powers and Roots

Since a complex number $z = (x, y) = (x, 0) + (0, y)$
$= x + iy$

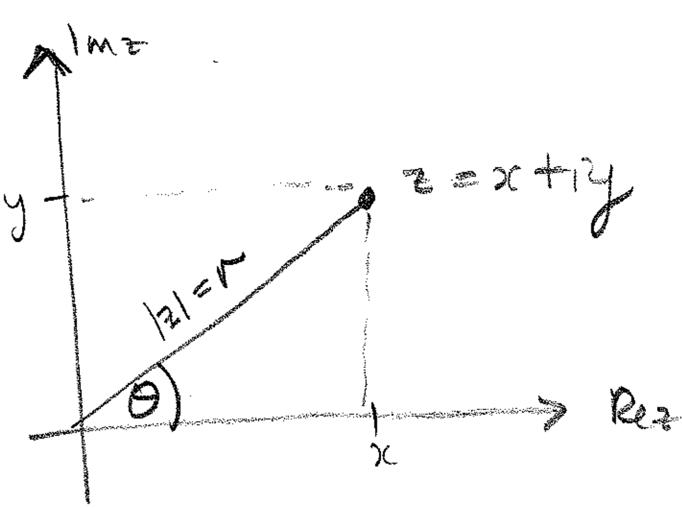
is an ordered pair of real nos., it can be represented by the point (x, y) in the plane $(\mathbb{R}^2)$ using Cartesian coordinates.



The complex conjugate $\bar{z} = x - iy$ of $z = x + iy$ is obtained by reflecting the point (x, y) in the x-axis.



Now suppose instead we use polar coordinates.



Then

$$x = r\cos\theta \qquad , \quad z = x + iy = r(\cos\theta + i\sin\theta)$$
$$y = r\sin\theta$$

Also

$r = \sqrt{x^2+y^2} = |z|$, the absolute value of z.

$\theta = \arctan\left(\frac{y}{x}\right)$, called the <u>argument</u> of z and written $\arg z$. Measured counterclockwise from the positive real axis.

ie

$$\arg z := \arctan\left(\frac{y}{x}\right).$$

One problem with this is that the formula doesn't make sense if $z=0$ and in this case we say $\theta = \arg z$ is undefined.

Also have trouble if $x=0$ but $y \neq 0$, but this can be got round.

Finally, since going round a full circle gets us back where we started, $\theta = \arg z$ is only defined up to an integer multiple of 2π (an example of a multi-valued function).

A common way round this is to restrict the values of θ, e.g. to $-\pi < \theta \leq \pi$.

This is called taking a branch in this case the principal branch of the argument, written Arg z

$$-\pi < \operatorname{Arg} z \leq \pi.$$

and $$\arg z = \operatorname{Arg} z + 2n\pi, \quad n \in \mathbb{Z}.$$

e.g. If $z = 1+i$, then

$$|z| = \sqrt{1^2+1^2} = \sqrt{2}$$

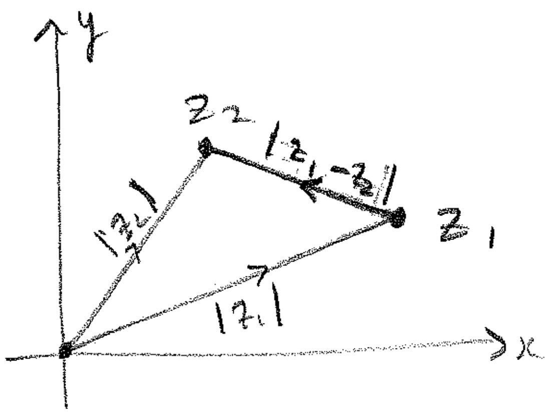
$$\operatorname{Arg} z = \arctan\left(\frac{1}{1}\right) = \arctan(1) = \frac{\pi}{4}$$

$$\left(-\pi < \frac{\pi}{4} \leq \pi\right)$$

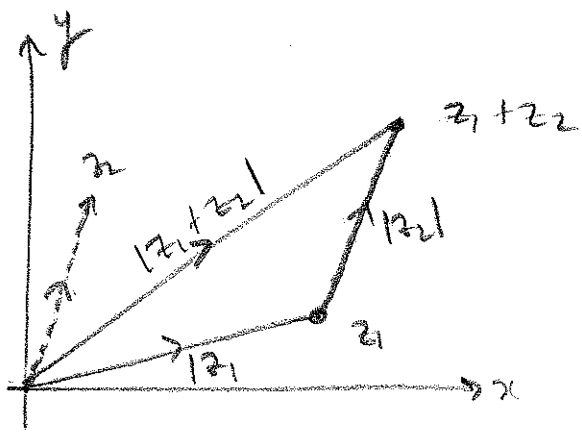
and so

$$1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$$ in polar form.

Treating Complex Numbers like Vectors - the Triangle Inequality



$|z_1 - z_2|$ measures the distance between z_1 & z_2



Each side of the triangle has length less than or equal to the sum of the lengths of the other two sides.

Hence $|z_1 + z_2| \leq |z_1| + |z_2|$

- triangle inequality.

Also have

$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$

- generalized

Multiplication and Division in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be two cplx nos. in polar form.

Then

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)\, r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$+ i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))$$

Recall the compound angle formulae from trigonometry

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$
$$\sin(a+b) = \sin a\cos b + \cos a\sin b.$$

Then

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

Hence

$$|z_1 z_2| = |z_1|\,|z_2| \qquad \left(\text{remember } \begin{matrix} r_1 = |z_1| \\ r_2 = |z_2| \end{matrix}\right)$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2 \qquad \left(\begin{matrix}\text{up to integer} \\ \text{multiples of } 2\pi\end{matrix}\right).$$

One can do a similar calculation for the quotient $\frac{z_1}{z_2}$ $(z_2 \neq 0)$ to get that

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$$

so that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \qquad \left(\begin{matrix}\text{up to integer multiples} \\ \text{of } 2\pi\end{matrix}\right)$$

Ex. $z_1 = -2+2i, \quad z_2 = 3i$

Check (ex). $z_1 z_2 = -6-6i, \quad \frac{z_1}{z_2} = \frac{2}{3} + \left(\frac{2}{3}\right)i$

By looking we see that

$$\operatorname{Arg} z_1 = \frac{3\pi}{4}, \quad \operatorname{Arg} z_2 = \frac{\pi}{2}$$

$$\operatorname{Arg} z_1 z_2 = -\frac{3\pi}{4} = \frac{3\pi}{4} + \frac{\pi}{2} - 2\pi$$

$$= \operatorname{Arg} z_1 + \operatorname{Arg} z_2 - 2\pi$$

$$\operatorname{Arg} z_1/z_2 = \frac{\pi}{4} = \frac{3\pi}{4} - \frac{\pi}{2} = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$$

Powers, de Moivre's Formula, Roots.

Let $z = r(\cos\theta + i\sin\theta)$ be in polar form.

Then from above

$$z^2 = r^2(\cos(\theta+\theta) + i\sin(\theta+\theta))$$
$$= r^2(\cos(2\theta) + i\sin(2\theta))$$

$$z^3 = z^2 z = r^2(\cos(2\theta) + i\sin(2\theta))$$
$$\cdot\, r(\cos(\theta) + i\sin(\theta))$$
$$= r^3(\cos(2\theta+\theta) + i\sin(2\theta+\theta))$$

$= r^3 (\cos(3\theta) + i\sin(3\theta))$

In general, by induction

$$\boxed{z^n = r^n (\cos(n\theta) + i\sin(n\theta))}$$

- de Moivre's Formula

- works for any integer
- gives us a very quick way of calculating powers of a given cplx. no.

Now suppose we want to go the other way - ie. find nth roots instead of nth powers of z.

ie. we want to find (all) w for which

$$w^n = z$$

(call all such w the nth roots of z).

Let $z = r(\cos\theta + i\sin\theta)$

and $w = s(\cos\varphi + i\sin\varphi)$

We want s, φ in terms of r, θ.

By de Moivre,

$$s^n(\cos(n\varphi) + i\sin(n\varphi)) = r(\cos\theta + i\sin\theta)$$

Thus $s^n = r$

$\Rightarrow$ $s = \sqrt[n]{r}$ [not the ordinary nth root of a positive real number]

while

$$n\varphi = \theta + 2k\pi, \qquad k \in \mathbb{Z}$$

ie $$\varphi = \frac{\theta}{n} + \frac{2k\pi}{n}, \qquad k \in \mathbb{Z}.$$

If we let k range from 0 to $n-1$ (inclusive), we get n distinct values of φ. If we take a larger range of values of k, we start to get repetition (i.e. values of φ which differ by an integer multiple of 2π).

Hence, there are n distinct solutions to $w^n = z$ $(z \neq 0)$ and they are given by

$$w = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right),$$

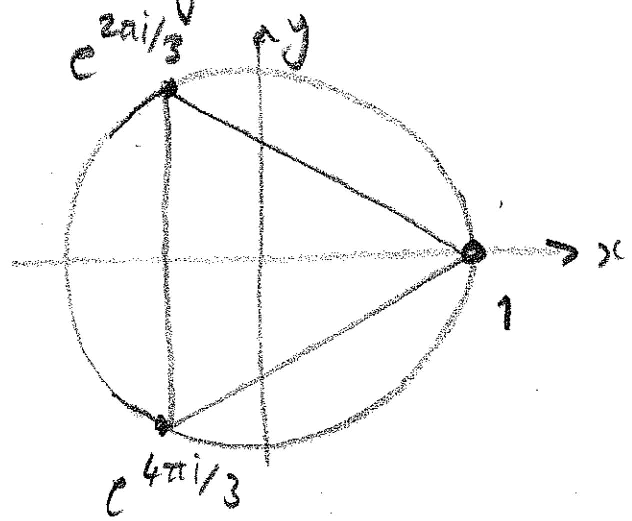
$$k = 0, \dots, n-1.$$

These n values are spaced equally round a circle of radius $\sqrt[n]{r}$.

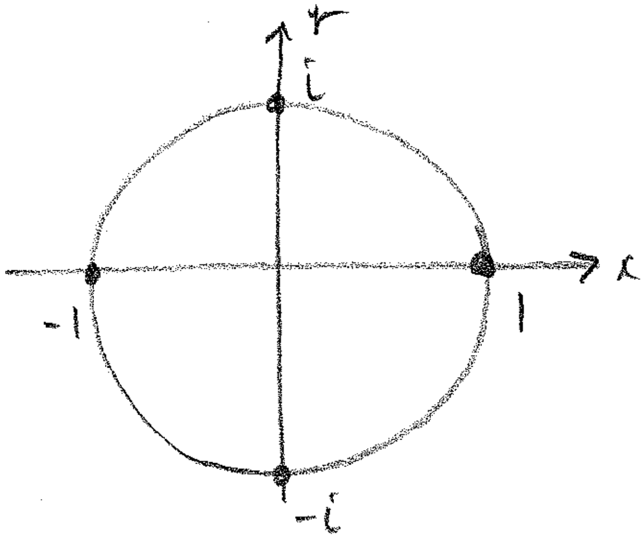
e.g. The cube roots of 1 are

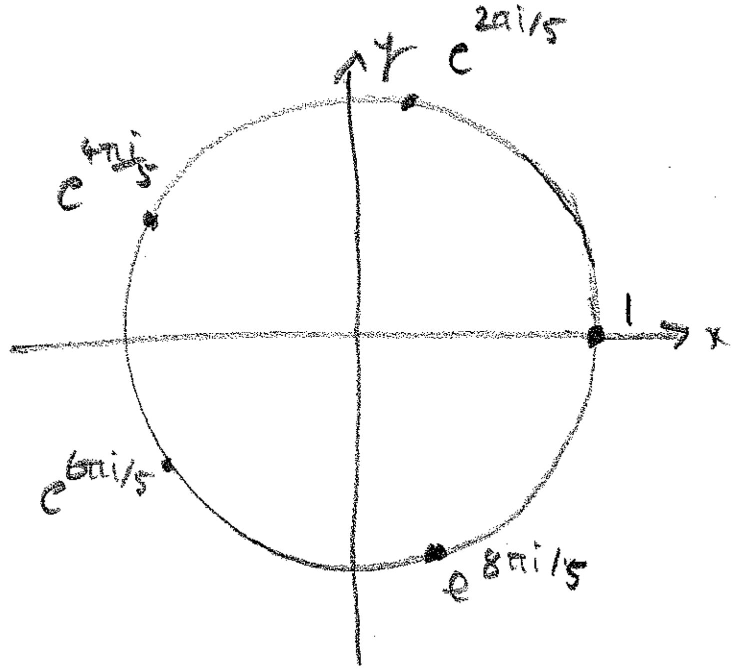
$1, \ e^{2\pi i/3}, \ e^{4\pi i/3}$ $\qquad (r = 1, \theta = 0)$

and they lie on an equilateral triangle



Similarly for fourth, fifth roots etc.





$z^4 = 1$

solns. $1, i, -1, -i$

$z^5 = 1$

solns. $1, e^{2\pi i/5}, e^{4\pi i/5}, e^{6\pi i/5}, e^{8\pi i/5}$