13 Complex Numbers

13.1 Complex Numbers, Complex Plane

The need for complex numbers arose in the middle ages when people wanted to solve polynomial equations like

$$x^2 = -1.$$

Formally a complex number z is an ordered pair (x, y) of real numbers.

Write

$$z = (x, y)$$

x is called the real part of z

y is called the imaginary part of z.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

Two very important operations on complex nos.–
addition and multiplication

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ be cplx. nos.

Define

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) := (x_1 + y_1, x_2 + y_2)$$

(usual addition of vectors in $\mathbb{R}^2$)

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

Note that in particular

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

$$(x_1, 0)(x_2, 0) = (x_1 x_2, 0).$$

ie Complex numbers of the form $(x, 0)$ 'behave' like real numbers and if we make the identification

$$(x, 0) = x,$$

we can say that the real numbers 'sit inside' the complex numbers or that the complex numbers extend the real numbers.

A very important complex number is

$$i := (0, 1)$$

Note that

$$
\begin{aligned}
i^2 &= (0,1)(0,1) \\
&= (0^2 - 1^2, 0 \times 1 + 0 \times 1) \\
&= (-1, 0) \\
&= -1.
\end{aligned}
$$

i is called 'imaginary i'. Also called j by electrical engineers.

Easy to check the following

$$
\begin{aligned}
(x, y) &= (x, 0) + (0, y) \\
&= x + (0,1)(y, 0) \\
&= x + iy.
\end{aligned}
$$

i.e. any complex number $z=(x,y)$ can be written (uniquely) as

$$z = x + iy$$

which is the usual way of writing cplx nos. The set of all complex numbers $z=x+iy$ is written as $\mathbb{C}$ (like $\mathbb{R}$ for the real numbers).

Ex. $z_1 = 8+3i$, $z_2 = 9-2i$.

$\operatorname{Re} z_1 = 8$, $\operatorname{Im} z_1 = 3$

$\operatorname{Re} z_2 = 9$, $\operatorname{Im} z_2 = -2$

$$z_1 + z_2 = 8+9+(3-2)i = 17+i$$

$$\begin{aligned} z_1 z_2 &= (8+3i)(9-2i) \\ &= (72 - 16i + 27i + 3(-2)i^2) \\ &= 72 - 16i + 27i + -6(-1) \\ &= 78 + 11i. \end{aligned}$$

Subtraction Division

Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in \mathbb{C}$.

Define

$$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$$

and if $z_2 \neq 0$ and we set

$$w := \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2},$$

one can check that

$$w z_2 = z_1$$

and so we can use the above formula to define $\frac{z_1}{z_2}$.

Note the special case for $z = x + iy$

$$\frac{1}{z} := \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

e.g. $z_1 = 1+2i$, $z_2 = 3+4i$

$$z_1 - z_2 = 1-3 + (2-4)i = -2-2i$$

$$\frac{z_1}{z_2} = \frac{(1+2i)(3-4i)}{3^2+4^2} = \frac{3-(-8)+(-4+6)i}{25}$$

$$= \frac{11+2i}{25}$$

$$= \frac{11}{25} + \frac{2}{25} i$$

Two Other Operations

Complex Conjugation $\bar{z} = \overline{x+iy} := x - iy$

Absolute Value or Modulus $|z| = |x+iy| = \sqrt{x^2+y^2}$

Note: $z\bar{z} = (x+iy)(x-iy) = x^2-(-y^2) - ixy + ixy$

$$= x^2+y^2 = |z|^2$$

ie $z\bar{z} = |z|^2$

or $|z| = \sqrt{z\bar{z}}$

Also

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

which gives us another way of deriving the formula for dividing complex nos.

13.2 The Complex Plane, Polar Form, Powers and Roots

Since a complex number $z = (x, y) = (x, 0) + (0, y)$
$= x + iy$

is an ordered pair of real nos., it can be represented by the point (x, y) in the plane $(\mathbb{R}^2)$ using Cartesian coordinates.