§ 1.5 Linear ODEs

Bernoulli Equation

Population Dynamics

A first order ODE is <u>linear</u> if it can be written in the form

$$y' + p(x)y = r(x).$$

e.g. $y'\cos x + y\sin x = x$ is linear as we can divide by $\cos x$ to get

$$y' + y\tan x = \frac{x}{\cos x}$$

(may have to worry about places where $\cos x = 0$).

Note that

$$y' + p(x)y = r(x)$$

is linear both as a fn of y and of y'.

In engineering, $r(x)$ is called the <u>input</u> or <u>forcing</u> applied to the system and y is the <u>output</u> or <u>response</u> of the system to this input.

We want to solve $y' + p(x)y = r(x)$ on some interval $J = (a, b)$. The simplest case is when the rhs $r(x)$ is identically 0 and we have the <u>homogeneous</u> eqn.

$$y' + p(x)y = 0$$

Separating variables gives

$$\frac{dy}{y} = -p(x)dx$$

Now integrate

$$\int \frac{dy}{y} = \int -p(x)dx$$

$$\ln|y| = -\int p(x)dx + k$$

$$|y| = e^{k - \int p(x)dx}$$

$$y = c\, e^{-\int p(x)dx}, \quad c = \pm e^k$$

($\pm$ depending on whether $y \geq 0$ or $y \leq 0$)

May also have $c = 0$ in which case we have the trivial solution $y = 0$.

We now consider the case where $r(x)$ is not identically zero on J.

In this case the ODE

$$y' + p(x)y = r(x)$$

is called inhomogeneous (nonhomogeneous)

First we separate the variables as before

$$\frac{dy}{dx} + p(x)y = r(x)$$

$$dy + p(x)y\,dx = r(x)\,dx$$

$$(p(x)y - r(x))\,dx + dy = 0$$

This is of course not as simple as last time, but we can luckily find an integrating factor depending only on x.

In the terminology of § 1.4,

$$P = p(x)y - r(x), \quad Q = 1$$

and $$R = \frac{1}{Q}(P_y - Q_x) = p(x).$$

As this depends only on x, by Thm 1, $\exists$ an integrating factor F depending only on x which satisfies

$$\frac{1}{F}\frac{dF}{dx} = R = p(x).$$

and by Thm 1 again we have

$$F(x) = e^{\int R(x)\,dx}$$

$$= e^{\int p(x)\,dx}$$

Now multiply both sides of our ODE by $F(x)$.

$$e^{\int p(x)\,dx} y' + e^{\int p(x)\,dx} p(x) y = e^{\int p(x)\,dx} r(x).$$

$$\frac{d}{dx}\left(e^{\int p(x)\,dx} y\right) = e^{\int p(x)\,dx} r(x)$$

by the product rule (used backwards).

Now integrate (again) to get.

$$e^{\int p(x)\,dx} y = \int e^{\int p(x)\,dx} r(x)\,dx + c$$

Divide by $e^{\int p(x)\,dx}$ to get

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} r(x)\,dx + c\right)$$

or, if we let $h = \int p(x)dx$, then

$$y = e^{-h}\left(\int e^{h} r\,dx + c\right)$$

n.b. as $e^{h} = e^{\int p(x)dx}$ is just an integrating factor, we don't need to bother about the const of integration in $h = \int p(x)dx$ and can just take it to be 0 (or anything else we like).
In any case, one can see from the form of the sol^n that any change in the const of integration would cancel out because of the e^{-h} and the e^{h} and so we get the same sol^n anyway.

If we write our sol$^{\Delta}$ as

$$y = e^{-h} \int e^{h} r \, dx + c e^{-h},$$

we recognise $c e^{-h}$ as the sol^{n} to the homogeneous eq$^{\Delta}$ $y' + p(x) y = 0$.

The other part depends on the rhs r (and would be 0 if r was 0 on J).

We can express this in engineering terms as

Total Output = Response to the Input r

$+$

Response to the Initial Data.

Ex.
$$y' - y = e^{2x}$$

Sol^n: Here $p = -1$ $\quad r = e^{2x}$

$h = \int p\,dx = -x$ (remember, const of integration $= 0$)

Thus the gen sol$^{\underline{n}}$ is

$$y = e^{-h}\left(\int e^{h} r(x)\,dx + c\right)$$

$$= e^{x}\left(\int e^{-x} e^{2x}\,dx + c\right)$$

$$= e^{x}\left(\int e^{x}\,dx + c\right)$$

$$= e^{x}\left(e^{x} + c\right)$$

$$= ce^{x} + e^{2x}$$

Can also solve this just by multiplying by the integrating factor $e^h = e^{-x}$. Get

$$(y' - y)e^{-x} = e^{2x}e^{-x}$$

$$\frac{d}{dx}(ye^{-x}) = e^x$$

Integrate.

$$ye^{-x} = \int e^x dx$$

$$= e^x + c$$

$$y = e^{2x} + ce^x$$

Ex. $y' + y \tan x = \sin 2x \quad , \quad y(0) = 1.$

Soln. Here $p = \tan x \quad , \quad r = \sin 2x = 2 \sin x \cos x$

$$\int p\,dx = \int \frac{\sin x}{\cos x}\,dx \qquad \begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}$$

$$= \int -\frac{du}{u}$$

$$= -\ln|u|$$

$$= \ln \frac{1}{|u|}$$

$$= \ln \left|\frac{1}{\cos x}\right|$$

$$= \ln|\sec x|$$

and

$$e^h = e^{\ln|\sec x|}$$

$$= |\sec x|$$

In fact, we can just take $e^h = \sec x$ as changing an integrating factor by a constant factor (± 1) still gives us an integrating factor.

Then

$$e^{-h} = \frac{1}{\sec x} = \cos x$$

and

$$e^{h} r = \sec x \cdot 2 \sin x \cos x$$

$$= 2 \sin x$$

and the general sol$^{\underline{n}}$ of the ODE is

$$y = e^{-h}\left(\int e^{h} r \, dx + c\right)$$

$$= \cos x \left(\int 2 \sin x \, dx + c\right)$$

$$= \cos x \left(-2 \cos x + c\right)$$

$$= c \cos x - 2 \cos^2 x .$$

Since $y(0) = 1$

$$1 = c \cdot 1 - 2 \cdot 1^2$$

$\Rightarrow c = 3$ and so the sol$^{\underline{n}}$ of the IVP is

$$y = 3 \cos x - 2 \cos^2 x .$$

Reduction to Linear Form - Bernoulli Equation

A Bernoulli equation is an ODE of the form

$$y' + p(x)y = g(x)\,y^{a}, \qquad a \in \mathbb{R}.$$

If $a = 0$ or 1, the ODE is linear.
Otherwise it is nonlinear. In this case we set

$$u(x) = \big(y(x)\big)^{1-a}$$

Differentiate and substitute for y' from the ODE to get

$$
\begin{aligned}
u' &= (1-a)\,y^{-a}\,y' \\
&= (1-a)\,y^{-a}\,(g y^{a} - p y) \\
&= (1-a)\,(g - p y^{1-a}) \\
&= (1-a)\,(g - p u) \qquad \text{as } u = y^{1-a} \\
&= (1-a)g - (1-a)pu.
\end{aligned}
$$

Thus u satisfies the linear ODE

$$u' + (1-a)pu = (1-a)pg.$$

This can then be solved for u using the same method as before. Then use $y = u^{\frac{1}{1-a}}$ to find y.

Ex. An example from population dynamics - the Logistic Equation.

$$y' = Ay - By^2$$

Soln Rewrite this as a Bernoulli eqn.

$$y' - Ay = -By^2$$

Here $a = 2$, so $u = y^{1-a} = y^{-1} = \frac{1}{y}$

Here

$$u' = -\frac{1}{y^2} y'$$

$$= -\frac{1}{y^2} (Ay - By^2)$$

$$= -\frac{A}{y} + B$$

$$= -Au + B$$

So $$u' + Au = B$$

Here $p = A$, $r = B$

$h = \int p\,dt = At$, $e^h = e^{At}$

and the general sol^n is

$$u = e^{-h}\left(\int e^h r\,dt + c\right)$$

$$= e^{-At}\left(\int e^{At} \cdot B\,dt + c\right)$$

$$= e^{-At}\left(\frac{B}{A} e^{At} + c\right)$$

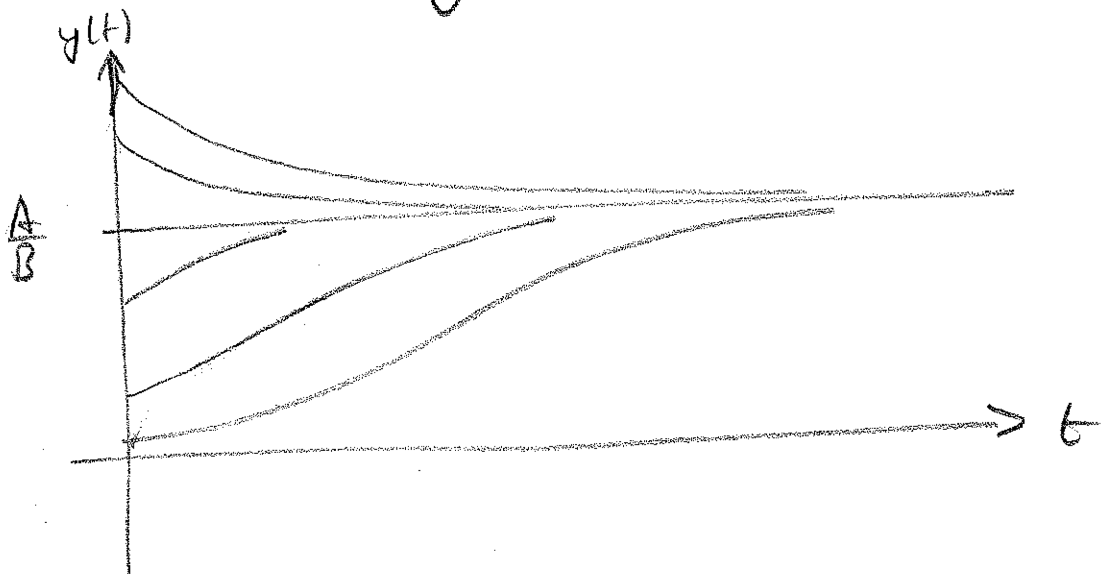
So

$$u = ce^{-At} + \frac{B}{A}$$

and since $u = \frac{1}{y}$, we have

$$y = \frac{1}{u} = \frac{1}{ce^{-At} + \frac{B}{A}}$$

Note also that $y \equiv 0$ is also a sol^{n} of $y' = Ay - By^2$ as can be seen directly from the ODE.



If $A > 0$, then because of the negative exponential, the population value tends to $\frac{1}{\frac{B}{A}} = \frac{A}{B}$.

Note that if $B = 0$,

$$y = \frac{1}{Ce^{-At}} = \frac{1}{C} e^{At}$$

- population grows without bound.
- Malthusian growth - models populations growing in an environment with infinite resources

The term $-By^2$ is a 'braking term' which prevents the population from growing without bound.

If we write

$$y' = Ay - By^2$$

$$= Ay\left(1 - \frac{B}{A} y\right)$$

then we see that $y' > 0$ if $y < \frac{A}{B}$ and $y' < 0$ if $y > \frac{A}{B}$.

$\Rightarrow$ y is increasing if $y < \frac{A}{B}$

and y is decreasing if $y > \frac{A}{B}$.

Also $y = \frac{A}{B}$ is another constant solution.

Note that t does not appear explicitly in the ODE $y' = Ay - By^2$.

An ODE of the form

$$y' = f(y)$$

in which t does not appear explicitly is called autonomous.

If $f(y_0) = 0$ for some y_0, then

$$y' = f(y_0) = 0$$

and $y = y_0$ is a constant solution.

In this case the point y_0 is called a critical point of the ODE.

A critical point is <u>stable</u> if solutions close to it for some t remain close to it for all further t.

A critical point is <u>unstable</u> if solutions close to it at some t do not remain close to it.

In the logistic example, we have two critical points, $y=0$ and $y=\frac{A}{B}$.

$y=0$ is unstable, while $y=\frac{A}{B}$ is stable.

One can also see this in a phase line plot where one makes little at many pts whose slope is y'.