MTH 362:Fall 2005
TEST II

Name: ______________________________

Note: You will not be given full credit if you only give the final answer. Show as much detail as you can.

§1.1

1. Given the differential equation $x^2y'' - 3xy' + 4y = 0$, $x > 0$.

(a) (10 pts) Show that $y_1 = x^2$ and $y_2 = x^2 \ln x$ are solutions to the given ODE.

$y_1' = 2x \qquad y_2' = 2x\ln x + x$

$y_1'' = 2 \qquad y_2'' = 2\ln x + 3$

$y_1: \quad x^2 \cdot 2 - 3x \cdot 2x + 4 \cdot x^2 = 2x^2 - 6x^2 + 4x^2 = 0$

$y_2: \quad x^2[2\ln x + 3] - 3x[2x\ln x + x] + 4x^2\ln x = 2x^2\ln x + 3x^2 - 6x^2\ln x - 3x^2 + 4x^2\ln x = 0$

§2.1

(b) (5 pts) Show that y_1 and y_2 are linearly independent solutions.

$$\frac{y_2}{y_1} = \frac{x^2\ln x}{x^2} = \ln x \neq \text{CONSTANT}$$

OR

$0 = ay_1 + by_2 = ax^2 + bx^2\ln x = x^2(a + b\ln x)$

$\Rightarrow \quad x^2 = 0 \quad$ or $\quad a + b\ln x = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad \ln x = -\frac{a}{b}$

$x = e^{-\frac{a}{b}}$

$\therefore$ lin. indep since x is not a constant

§1.6 P39 #15

2. (10 pts) Solve the initial value problem $\begin{cases} y' + 4y = 20 \\ y(0) = 2, \end{cases}$

$e^{4x}y' + 4e^{4x}y = 20e^{4x}$

$\frac{d}{dx}(ye^{4x}) = 20e^{4x}$

$ye^{4x} = 5e^{4x} + c$

$y(0) = 2 \Rightarrow 2 = 5 + c \Rightarrow c = -3$

$ye^{4x} = 5e^{4x} - 3$

$y = 5 - 3e^{-4x}$

$\frac{dy}{dx} = y' = 20 - 4y = 4(5 - y)$

$\frac{dy}{y-5} = -4$

$\ln|y-5| = -4x + c$

$|y-5| = e^{-4x+c}$

$y - 5 = ce^{-4x}$

$y = 5 + ce^{-4x}$

$y(0) = 2 \Rightarrow c = -3$

$y = 5 - 3e^{-4x}$

§1.3

P18 #8 3. (10 pts) Solve the differential equation $xy' = x + y$ by using the subsitution $u = \frac{y}{x}$.

$y = ux$
$y' = u'x + u$

$x^2u' + xu = x + ux$
$x^2u' = x$
$u' = \frac{1}{x}$
$u = \ln|x| + c$
$\frac{y}{x} = \ln|x| + c$
$y = x\ln|x| + cx$

CHECK $y' = 1 + \ln|x| + c$
$x + x\ln|x| + cx = x + x\ln|x| + cx$ ✓

§1.5 4. Consider the equation $(4x^3y^3 - 2xy)dx + (3x^4y^2 - x^2 + 2y)dy = 0$.

(a) (5 pts) Show that the equation is exact.

$\frac{\partial M}{\partial y} = 12x^3y^2 - 2x = \frac{\partial N}{\partial x}$

(b) (10 pts) Solve for the general solution of the equation.

$\int M\,dx = x^4y^3 - x^2y + f(y)$
$\int N\,dy = x^4y^3 - x^2y + y^2 + g(x)$ $\Rightarrow f(y) = y^2,\ g(x) = 0$
$x^4y^3 - x^2y + y^2 = C$

§2.1 5. (10 pts) A differential equation and one of its solution is given, apply the method of reduction of order to obtain another linearly independent solution.

$$x^4y'' + 2x^3y' - y = 0, \quad x > 0, \quad y_1 = e^{1/x}$$

$y_2 = ue^{\frac{1}{x}}$
$y_2' = u'e^{1/x} - \frac{u}{x^2}e^{\frac{1}{x}}$
$y_2'' = u''e^{1/x} - \frac{2u'}{x^2}e^{1/x} + ue^{\frac{1}{x}}\left[\frac{1}{x^4} + \frac{2}{x^3}\right]$
$x^4\left[u''e^{\frac{1}{x}} - \frac{2u'}{x^2}e^{\frac{1}{x}} + \frac{u}{x^4}e^{\frac{1}{x}} + \frac{2u}{x^3}e^{\frac{1}{x}}\right] + 2x^3\left[u'e^{\frac{1}{x}} - \frac{u}{x^2}e^{\frac{1}{x}}\right] - ue^{\frac{1}{x}} = 0$
$u''e^{\frac{1}{x}}[x^4] + u'e^{\frac{1}{x}}[-2x^2 + 2x^3] + ue^{\frac{1}{x}}[\cancel{1 + 2x - 2x - 1}] = 0$
$x^2e^{\frac{1}{x}}[u''x^2 + u'(2x - 2)] = 0$
$U' + u\left(\frac{2}{x} - \frac{2}{x^2}\right) = 0$
$\frac{U'}{U} = \frac{2}{x^2} - \frac{2}{x}$
$\ln|U| = -\frac{2}{x} - 2\ln|x|$

$U = \frac{1}{x^2}e^{-\frac{2}{x}}$
$u = \int U = \frac{1}{2}e^{-\frac{2}{x}}$
$y_2 = ue^{\frac{1}{x}} = \frac{1}{2}e^{-\frac{1}{x}}$

§1.6
P39 #37 6. (10 pts) Given the Bernoulli equation
&S1.3

$$\frac{dy}{dx} + 2xy = -xy^4$$

use the subsitution $v = y^{-3}$ to solve for its general solution. (express your answer in terms of y).

$v' = -3y^{-4}y' = -3y^{-4}(-xy^4 - 2xy) = 3x + 6xy^{-3} = 3x + 6xv$

$v' - 6xv = 3x$

$e^{-3x^2}(v' - 6xv = 3x)$

$\frac{d}{dx}(v e^{-3x^2}) = 3x e^{-3x^2}$

$v e^{-3x^2} = -\frac{1}{2} e^{-3x^2} + c$

$v = c e^{3x^2} - \frac{1}{2}$

$y = \sqrt[3]{\frac{1}{ce^{3x^2} - \frac{1}{2}}}$

$v' = 3x + 6xv = 3x(1+2v)$

$\frac{v'}{1+2v} = 3x$

$\frac{1}{2} \ln|1+2v| = \frac{3x^2}{2} + c$

$\ln|1+2v| = 3x^2 + 2c$

$|1+2v| = e^{3x^2}e^{2c} = c_1 e^{3x^2}$

$y^{-3} = v = \frac{c_1 e^{3x^2} - 1}{2}$

$y = \sqrt[3]{\frac{2}{c_1 e^{3x^2} - 1}}$

§1.4 7. In a certain culture of bacteria the rate of increase is proportional to the number present

(a) (10 pts) If there are 10^4 at the end of 3 hours and $4 \cdot 10^4$ at the end of 5 hours how many were there at the beginning?

$y' = ky$

$\frac{y'}{y} = k$

$\ln|y| = kx + c_1$

$|y| = e^{kx}e^{c_1}$

$y = c e^{kx}$

$y(3) = 10^4 = c e^{3k}$

$y(5) = 4 \cdot 10^4 = c e^{5k}$

$4 = e^{2k}$

$\ln 4 = 2k$

$k = \frac{\ln 4}{2} = \ln 4^{1/2} = \ln 2$

$y = c e^{x \ln 2}$

$y = c 2^x$

$y = \frac{10^4}{8} 2^x$

$y_0 = \frac{10^4}{8} = 1250$

(b) (5 pts) If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?

$y_0 e^{4k} = 2y_0$

$e^{4k} = 2$

$4k = \ln 2$

$k = \frac{\ln 2}{4}$

$y = y_0 e^{12 \frac{\ln 2}{4}}$

$= y_0 e^{3 \ln 2}$

$= y_0 e^{\ln 2^3}$

$= 8 y_0$

There will be 8 times the starting amount.

§1.7

8. An R–L circuit consists of a 100 volt DC battery connected in series with a 2 henry inductor and a 6 ohm resistor.

   (a) (5 pts) Use Kirchhoff's law to write the initial value problem, assume current starts to flow when the open switch is loced.

$$2\frac{dI}{dt} + 6I = 100$$

$$\frac{dI}{dt} + 3I = 50$$

   (b) (5 pts) Verify that $I = \frac{50}{3}(1 - e^{-3t}), t \geq 0$ is the solution to the IVP in part (a).

$$I' = 50e^{-3t}$$

$$2[50e^{-3t}] + 6\left[\tfrac{50}{3}(1 - e^{-3t})\right] \stackrel{?}{=} 100$$

$$100e^{-3t} + 100 - 100e^{-3t} \stackrel{?}{=} 100 \quad \checkmark$$

   (c) (5 pts) At what time t does the current $I(t)$ reach 99% of its steady state value?

$$1 - e^{-3t} = .99$$

$$e^{-3t} = .01 = \frac{1}{100}$$

$$-3t = -\ln 100$$

$$3t = \ln 100$$

$$t = \frac{\ln 100}{3} = 1.535$$