4.3 Linearly Independent Sets; Bases

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $c_1 = 0, \ldots, c_p = 0$.

The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

The following results from Section 1.7 are still true for more general vectors spaces.

A set containing the zero vector is linearly dependent.

A set of two vectors is linearly dependent if and only if one is a multiple of the other.

A set containing the zero vector is linearly independent.

EXAMPLE: $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & 0 \end{bmatrix}\right\}$ is a linearly ___________________ set.

EXAMPLE: $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix}\right\}$ is a linearly ___________________ set since $\begin{bmatrix} 3 & 6 \\ 9 & 11 \end{bmatrix}$ is not a multiple of $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem 4

An indexed set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some vector $\mathbf{v}_j$ $(j > 1)$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

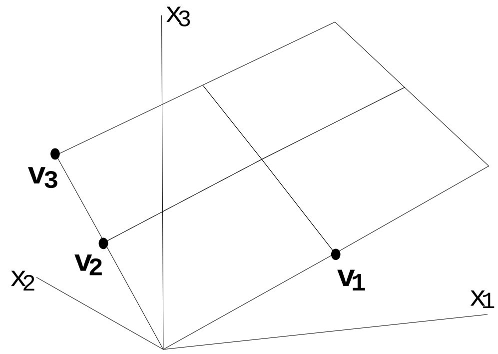
EXAMPLE: Let $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ be a set of vectors in $\mathbf{P}_2$ where $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$, and $\mathbf{p}_3(t) = 4t + 2t^2$. Is this a linearly dependent set?

Solution: Since $\mathbf{p}_3 =$ _____$\mathbf{p}_1 +$ _____$\mathbf{p}_2$, $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a linearly _____________________ set.

A Basis Set

Let H be the plane illustrated below. Which of the following are valid descriptions of H?

(a) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$
(c) $H = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



A *basis set* is an "efficient" spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ to both be examples of basis sets or bases (plural for basis) for H.

DEFINITION

Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

(i) β is a linearly independent set, and

(ii) $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$.

EXAMPLE: Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbf{R}^3$. The set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is called a **standard basis** for $\mathbf{R}^3$.

Solutions: (Review the IMT, page 129) Let

$A = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since A has 3 pivots, the columns of A are linearly

________________________ by the IMT and the columns of A ______________ __________

by IMT. Therefore, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a basis for $\mathbf{R}^3$.

EXAMPLE: Let $S = \{1,\ t,\ t^2,\ \ldots, t^n\}$. Show that S is a basis for $\mathbf{P}_n$.

Solution: Any polynomial in $\mathbf{P}_n$ is in span of S. To show that S is linearly independent, assume

$$c_0 \cdot 1 + c_1 \cdot t + \cdots + c_n \cdot t^n = \mathbf{0}$$

Then $c_0 = c_1 = \cdots = c_n = 0$. Hence S is a basis for $\mathbf{P}_n$.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$.

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a **basis** for $\mathbf{R}^3$?

Solution: Again, let $A = [\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3] = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$. Using row reduction,

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

and since there are 3 pivots, the columns of A are linearly independent and they span $\mathbf{R}^3$ by the IMT. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a **basis** for $\mathbf{R}^3$.

EXAMPLE: Explain why each of the following sets is **not** a basis for $\mathbf{R}^3$.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$

Bases for Nul A

EXAMPLE: Find a basis for Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Solution: Row reduce $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$:

$$\begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 &= -2x_2 - 13x_4 - 33x_5 \\ x_3 &= 6x_4 + 15x_5 \end{aligned}$$

x_2, x_4 and x_5 are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$\mathbf{u} \qquad\qquad \mathbf{v} \qquad\qquad \mathbf{w}$$

Therefore $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is a spanning set for Nul A. In the last section we observed that this set is linearly independent. Therefore $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ is a basis for Nul A. The technique used here always provides a linearly independent set.

The Spanning Set Theorem

A basis can be constructed from a spanning set of vectors by discarding vectors which are linear combinations of preceding vectors in the indexed set.

EXAMPLE: Suppose $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

Solution: If $\mathbf{x}$ is in Span$\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, then

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2)$$

$$= \_\_\_\_\_\_\_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\_\_\_\_\_\_\mathbf{v}_2$$

Therefore,

$$\text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1,\mathbf{v}_2\}.$$

THEOREM 5 The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S - say $\mathbf{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Bases for Col A

EXAMPLE: Find a basis for Col A, where

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution: Row reduce:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Note that

$\mathbf{b}_2 =$ ____$\mathbf{b}_1$ and $\mathbf{a}_2 =$ ____$\mathbf{a}_1$

$\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3$ and $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$

$\mathbf{b}_1$ and $\mathbf{b}_3$ are not multiples of each other

$\mathbf{a}_1$ and $\mathbf{a}_3$ are not multiples of each other

Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Therefore $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis for Col A.

THEOREM 6

The pivot columns of a matrix A form a basis for Col A.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Find a basis for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$ and note that Col $A =$ Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

By row reduction, $A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore a basis

for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $\left\{ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \right\}$.

Review:

1. To find a basis for Nul A, use elementary row operations to transform $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to an equivalent reduced row echelon form $\begin{bmatrix} B & \mathbf{0} \end{bmatrix}$. Use the reduced row echelon form to find parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. The vectors found in this parametric form of the general solution form a basis for Nul A.

2. A basis for Col A is formed from the pivot columns of A. **Warning: Use the pivot columns of A, not the pivot columns of B, where B is in reduced echelon form and is row equivalent to** A.