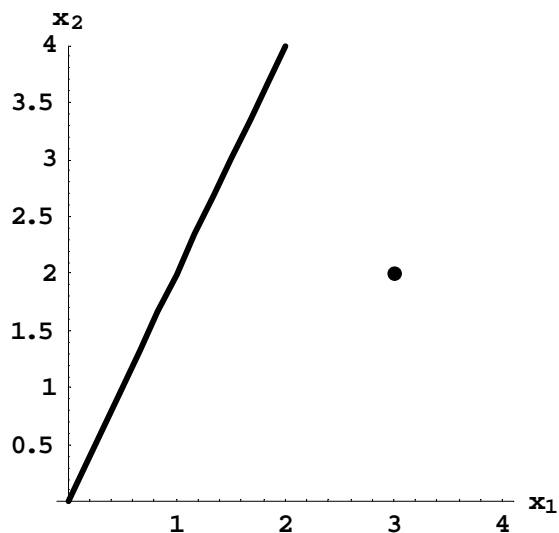


6.1 Inner Product, Length & Orthogonality

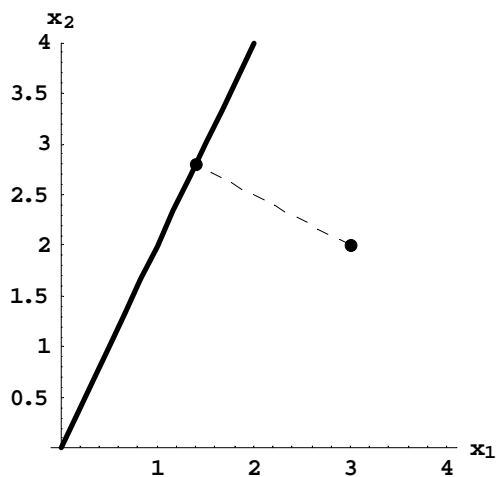
Not all linear systems have solutions.

EXAMPLE: No solution to $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ exists. Why?

$A\mathbf{x}$ is a point on the line spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and \mathbf{b} is not on the line. So $A\mathbf{x} \neq \mathbf{b}$ for all \mathbf{x} .



Instead find $\hat{\mathbf{x}}$ so that $A\hat{\mathbf{x}}$ lies "closest" to \mathbf{b} .



Using information we will learn in this chapter, we will find that $\hat{\mathbf{x}} = \begin{bmatrix} 1.4 \\ 0 \end{bmatrix}$, so that $A\hat{\mathbf{x}} = \begin{bmatrix} 1.4 \\ 2.8 \end{bmatrix}$.

Segment joining $A\hat{\mathbf{x}}$ and \mathbf{b} is *perpendicular* (or *orthogonal*) to the set of solutions to $A\mathbf{x} = \mathbf{b}$.

Need to develop fundamental ideas of *length*, *orthogonality* and *orthogonal projections*.

The Inner Product

Inner product or dot product of $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Note that

$$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + \cdots + v_n u_n = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \mathbf{u} \cdot \mathbf{v}$$

THEOREM 1

Let \mathbf{u}, \mathbf{v} and \mathbf{w} be vectors in \mathbf{R}^n , and let c be any scalar. Then

- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Combining parts b and c, one can show

$$(c_1 \mathbf{u}_1 + \cdots + c_p \mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})$$

Length of a Vector

For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, the **length** or **norm of \mathbf{v}** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

For example, if $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ (distance between $\mathbf{0}$ and \mathbf{v})

Picture:

For any scalar c ,

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|$$

Distance in \mathbf{R}^n

The **distance between \mathbf{u} and \mathbf{v}** in \mathbf{R}^n :

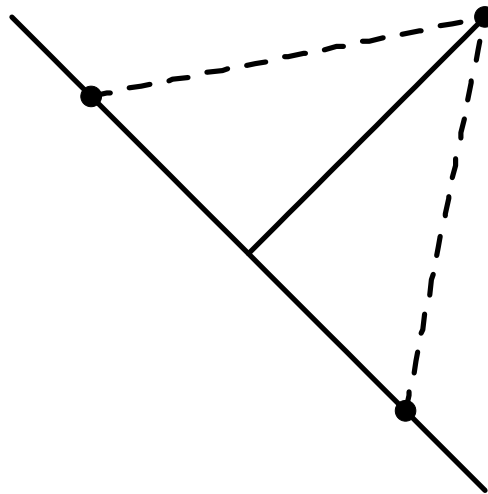
$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

This agrees with the usual formulas for \mathbf{R}^2 and \mathbf{R}^3 . Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

Then $\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2)$ and

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(u_1 - v_1, u_2 - v_2)\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

Orthogonal Vectors



$$\begin{aligned} [\text{dist}(\mathbf{u}, \mathbf{v})]^2 &= \|\mathbf{u} - \mathbf{v}\|^2 = (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (\mathbf{u}) \cdot (\mathbf{u} - \mathbf{v}) + (-\mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \\ &= \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + -\mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} \\ \Rightarrow [\text{dist}(\mathbf{u}, \mathbf{v})]^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} \end{aligned}$$

Similarly,

$$[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}$$

Since $[\text{dist}(\mathbf{u}, -\mathbf{v})]^2 = [\text{dist}(\mathbf{u}, \mathbf{v})]^2$, $\mathbf{u} \cdot \mathbf{v} = \underline{\hspace{2cm}}$.

Two vectors \mathbf{u} and \mathbf{v} are said to be **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$.

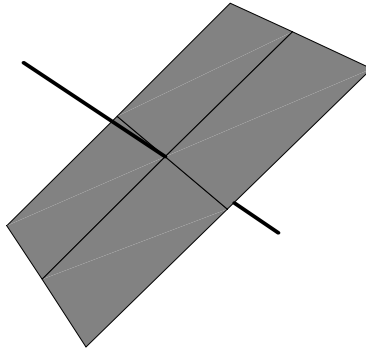
Also note that if \mathbf{u} and \mathbf{v} are orthogonal, then $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

THEOREM 2 THE PYTHAGOREAN THEOREM

Two vectors \mathbf{u} and \mathbf{v} are orthogonal if and only if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Orthogonal Complements

If a vector \mathbf{z} is orthogonal to every vector in a subspace W of \mathbf{R}^n , then \mathbf{z} is said to be **orthogonal to W** . The set of vectors \mathbf{z} that are orthogonal to W is called the **orthogonal complement** of W and is denoted by W^\perp (read as “ W perp”).



Row, Null and Columns Spaces

THEOREM 3

Let A be an $m \times n$ matrix. Then the orthogonal complement of the row space of A is the nullspace of A , and the orthogonal complement of the column space of A is the nullspace of A^T :

$$(\text{Row } A)^\perp = \text{Nul } A, \quad (\text{Col } A)^\perp = \text{Nul } A^T.$$

Why? (See complete proof in the text) Consider $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note that $A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and so \mathbf{x} is orthogonal to the row A since \mathbf{x} is orthogonal to $\mathbf{r}_1, \dots, \mathbf{r}_m$.

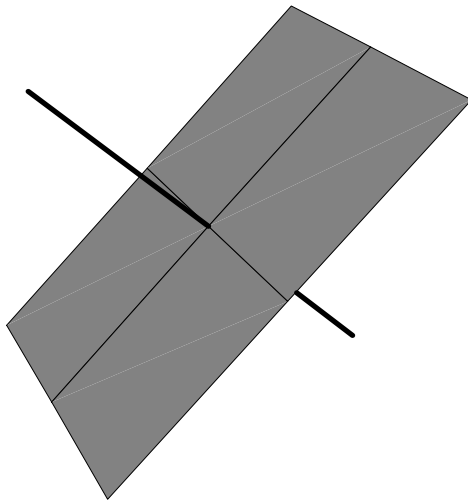
EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$.

Basis for $\text{Nul } A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and therefore $\text{Nul } A$ is a plane in \mathbf{R}^3 .

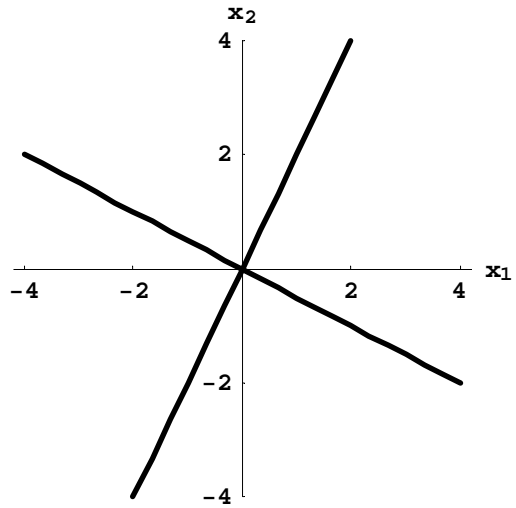
Basis for $\text{Row } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and therefore $\text{Row } A$ is a line in \mathbf{R}^3 .

Basis for $\text{Col } A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and therefore $\text{Col } A$ is a line in \mathbf{R}^2 .

Basis for $\text{Nul } A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and therefore $\text{Nul } A^T$ is a line in \mathbf{R}^2 .



Subspaces $\text{Nul } A$ and $\text{Row } A$



Subspaces $\text{Nul } A^T$ and $\text{Col } A$