2.3 Characterizations of Invertible Matrices

**Theorem 8 (The Invertible Matrix Theorem)**

Let \( A \) be a square \( n \times n \) matrix. The following statements are equivalent (i.e., for a given \( A \), they are either all true or all false).

- a. \( A \) is an invertible matrix.
- b. \( A \) is row equivalent to \( I_n \).
- c. \( A \) has \( n \) pivot positions.
- d. The equation \( Ax = 0 \) has only the trivial solution.
- e. The columns of \( A \) form a linearly independent set.
- f. The linear transformation \( x \rightarrow Ax \) is one-to-one.
- g. The equation \( Ax = b \) has at least one solution for each \( b \) in \( \mathbb{R}^n \).
- h. The columns of \( A \) span \( \mathbb{R}^n \).
- i. The linear transformation \( x \rightarrow Ax \) maps \( \mathbb{R}^n \) onto \( \mathbb{R}^n \).
- j. There is an \( n \times n \) matrix \( C \) such that \( CA = I_n \).
- k. There is an \( n \times n \) matrix \( D \) such that \( AD = I_n \).
- l. \( A^T \) is an invertible matrix.

**EXAMPLE:** Use the Invertible Matrix Theorem to determine if \( A \) is invertible, where

\[
A = \begin{bmatrix}
1 & -3 & 0 \\
-4 & 11 & 1 \\
2 & 7 & 3
\end{bmatrix}.
\]

**Solution**

\[
A = \begin{bmatrix}
1 & -3 & 0 \\
-4 & 11 & 1 \\
2 & 7 & 3
\end{bmatrix} \sim \cdots \sim \begin{bmatrix}
1 & -3 & 0 \\
0 & -1 & 1 \\
0 & 0 & 16
\end{bmatrix}
\]

3 pivot positions

**Circle correct conclusion:** Matrix \( A \) is / is not invertible.
EXAMPLE: Suppose $H$ is a $5 \times 5$ matrix and suppose there is a vector $v$ in $\mathbb{R}^5$ which is not a linear combination of the columns of $H$. What can you say about the number of solutions to $Hx = 0$?

Solution Since $v$ in $\mathbb{R}^5$ is not a linear combination of the columns of $H$, the columns of $H$ do not __________ $\mathbb{R}^5$.

So by the Invertible Matrix Theorem, $Hx = 0$ has __________________________.

Invertible Linear Transformations

For an invertible matrix $A$,

\[
A^{-1}A x = x \text{ for all } x \text{ in } \mathbb{R}^n \\
\text{and} \\
AA^{-1} x = x \text{ for all } x \text{ in } \mathbb{R}^n.
\]

Pictures:

A linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

\[
S(T(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n \\
\text{and} \\
T(S(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n.
\]

Theorem 9

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear transformation and let $A$ be the standard matrix for $T$. Then $T$ is invertible if and only if $A$ is an invertible matrix. In that case, the linear transformation $S$ given by $S(x) = A^{-1}x$ is the unique function satisfying

\[
S(T(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n \\
\text{and} \\
T(S(x)) = x \text{ for all } x \text{ in } \mathbb{R}^n.
\]