Chapter 1

System of Linear Equations

\[ a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \]
\[ \vdots \]
\[ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \]

Vector Equation

\[ \begin{bmatrix} a_{11} \\ \vdots \\ a_{mn} \end{bmatrix} x = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \]

Matrix Equation

\[ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \]

Solve such systems by row reducing the augmented matrix \([A \ 1 \ b]\) to reduced echelon form.
Echelon Forms

Echelon Form

1. All nonzero rows are above any rows of all zeroes.

2. Each leading entry of a row is in a column to the right of the leading entry in the row above it.

3. All entries in a column below the leading entry are 0.

\[
\begin{bmatrix}
\times & \times & \times \\
\times & \times & \times \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

Reduced Echelon Form

4. The leading entry (pivot) in each nonzero row is 1.

5. Each leading 1 is the only nonzero entry in its column.
Thm 1. The reduced echelon form of a matrix is unique in each matrix is row equivalent to one and only one reduced echelon matrix.

The standard (non-reduced) echelon form is not unique.

Thm 2. (p.24) A linear system $Ax = b$ is consistent iff the rightmost column of the reduced echelon form (r.e.f.) of the augmented matrix $[A | b]$ is not a pivot column.
Matrix Equations

\[ Ax = 0 \] has a non-trivial solution if and only if there are free variables.

\[ Ax = b \] has a solution if and only if \( b \) is a linear combination of the columns of \( A \).

Theorem 4 (p.43) A \( m \times n \) matrix, TFAE

a). \( \forall b \in \mathbb{R}^m \), \( Ax = b \) has a solution.

b). Each \( b \) in \( \mathbb{R}^m \) is a linear combination of the columns of \( A \).

c). The columns of \( A \) span \( \mathbb{R}^m \).

d). \( A \) has a pivot position in every row.

The general solution to \( Ax = b \) is obtained by adding one solution to \( Ax = b \) to the general solution to \( Ax = 0 \). (Theorem 6, p.53)
\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ is linear} \]

If \[ T(cu) = cT(u) \quad \forall \text{ vectors } u, v \]
\[ T(u + v) = T(u) + T(v) \quad \forall \text{ scalars } c. \]

The standard matrix of a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is

\[
A = \begin{bmatrix}
T(e_1) & T(e_2) \\
\vdots & \vdots \\
T(e_n) & T(e_n)
\end{bmatrix}
\]

where \( \{e_1, \ldots, e_n\} \) is the standard basis for \( \mathbb{R}^n \).
Chapter 2  Matrices

Inverses (only for square matrices) \( AA^{-1} = A^{-1}A = I_n \).

To find the inverse of an \( n \times n \) matrix \( A \), row reduce the augmented matrix

\[
\begin{bmatrix}
A & I_n
\end{bmatrix}
\]

to

\[
\begin{bmatrix}
I_n & A^{-1}
\end{bmatrix}
\]

in row reduce the l.h.s to \( I_n \) & the r.h.s is then automatically \( A^{-1} \).

Thm 6. \( A, B \) inv. \( n \times n \) matrices

a. \( A^{-1} \) inv. \( \Leftrightarrow (A^{-1})^{-1} = A \)

b. \( AB \) inv. \( \Leftrightarrow (AB)^{-1} = B^{-1}A^{-1} \)

c. \( A^T \) inv. \( \& (A^T)^{-1} = (A^{-1})^T \).
Theorem 8  Invertible Matrix Theorem (P.129).

A n x n matrix, TFAE

a. A is invertible, i.e., $\exists B$ s.t. $AB = BA = I_n$.

b. $A$ is row equivalent to $I_n$.

c. $A$ has $n$ pivot positions

d. $Ax = 0$ has only the trivial solution $x = 0$.

e. The cols. of $A$ are lin ind.

f. The linear transformation $x \mapsto Ax$ is one-to-one.

$Ax = b$ has at least one soln. $\forall b \in \mathbb{R}^n$.

h. The cols. of $A$ span $\mathbb{R}^n$.

i. The linear transformation $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

j. $\exists$ an n x n matrix $C$ s.t. $CA = I_n$.

k. $\exists$ an n x n matrix $D$ s.t. $AD = I_n$.

l. $A^T$ is invertible.
m. The cols. of $A$ form a basis of $\mathbb{R}^n$.

n. $\text{Col } A = \mathbb{R}^n$.

o. $\dim \text{Col } A = n$.

p. $\text{Rank } A = n$.

q. $\text{Nul } A = \{0\}$.

r. $\dim \text{Nul } A = 0$.

s. 0 is not an eigenvalue of $A$.

t. $\det A \neq 0$. 
Chapter 3  Determinants

A \text{nxn} matrix

A_{ij} minor matrix \((n-1) \times (n-1)\) obtained by blocking out the \(i\)-th row and \(j\)-th column of \(A\).

\[
\det A = (-1)^{i+j} \det A_{i1} + (-1)^{i+2} \det A_{i2} + \ldots + (-1)^{i+n} \det A_{in}
\]
- cofactor expansion along row \(i\)

\[
\begin{align*}
&= (-1)^{1+j} \det A_{1j} + (-1)^{2+j} \det A_{2j} + \ldots + (-1)^{n+j} \det A_{nj} \\
&\text{cofactor expansion along col.}\ j
\end{align*}
\]

\[A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{2\times2}\]

\[\det A = ad - bc.\]
If $A$ is upper or lower triangular, then $\det A$ is the product of the diagonal entries.

Cofactor expansion is computationally intensive. The best idea is to use row operations to make $A$ as nearly triangular as possible before computing $\det A$.

**Remark:** Elementary row operations leave $\det A$ unchanged except for swapping rows which changes the sign of $\det A$, and multiplying one row by a scalar factor which changes $\det A$ by this factor.

**Facts:**

\[
\begin{align*}
\det \ (AB) &= \det A \ \det B \\
\det \ A^T &= \det A \\
A^{-1} \Rightarrow \ &\det A^{-1} = \frac{1}{\det A}.
\end{align*}
\]
Chapter 4 Vector Spaces

Subspaces A subset $H$ of a vector space $V$ is a subspace of $V$ if

1. $0_V \in H$

2. $H$ is closed under addition - i.e. $u + v \in H \quad \forall u, v \in H$

3. $H$ is closed under scalar multiplication - i.e. $cu \in H \quad \forall u \in H, c \in \mathbb{R}$. (Defn of subspace).
Linear Independence - Bases

Let $V$ be a v.s. and $v_1, \ldots, v_p \in V$.

$\text{Span } \{v_1, \ldots, v_p\} = \{\text{all linear combs. of } v_1, \ldots, v_p\} = \{c_1v_1 + \cdots + c_pv_p, \ c_1, \ldots, c_p \in \mathbb{R}\}$.

$\text{Span } \{v_1, \ldots, v_p\}$ is a subspace of $V$.

If $\text{Span } \{v_1, \ldots, v_p\} = V$, we say $\{v_1, \ldots, v_p\}$ spans $V$.

$\{v_1, \ldots, v_p\}$ is linearly independent if

$$c_1v_1 + \cdots + c_pv_p = 0 \Rightarrow c_1 = c_2 = \cdots = c_p = 0 \text{ (only trivial solution)}.$$

If $\{v_1, \ldots, v_p\}$ spans $V$ & is lin. ind., we say it is a basis for $V$. 
Simple cases

1. A vector $v$ is li. ind. iff $v \neq 0$.

2. Two vectors $v, w$ are li. ind. iff neither vector is a scalar multiple of the other.

**Theorem 5** Spanning Set Theorem, p. 239.

Let $S = \{v_1, \ldots, v_p\}$ be a set in $V$ and let $H = \text{Span}\{v_1, \ldots, v_p\}$.

a. If one of the vectors in $S$ — say $v_i$ — is a li. comb. of the remaining vectors in $S$, then the set formed from $S$ by removing $v_i$ still spans $H$.

b. If $H \neq \{0\}$, some subset of $S$ is a basis for $H$.

Can use this to make an algorithm to extract a basis from $S$. Usually quicker to do it by making the vectors the columns of a matrix $A$ and then finding $\text{col} \ A$. 
Dimension.

If a v.s. $V$ is spanned by a finite set, we say $V$ is finite-dimensional. In this case the number of elements in any basis of $V$ is always the same (Theorem 10, p. 257) and is called the dimension of $V$. 
Nullspace, Column Space

A \text{ m x n matrix}

x \rightarrow Ax \text{ gives a linear transformation from } \mathbb{R}^n \text{ to } \mathbb{R}^m.

\text{Null } A = \{ x : Ax = 0 \} \text{ is a subspace of } \mathbb{R}^n.

\text{Col } A = \text{ span } \{ \text{ columns of } A \} \nonumber
= \{ b : Ax = b \text{ for some } x \in \mathbb{R}^n \}
\text{ is a subspace of } \mathbb{R}^m.

To find \text{Null } A, \text{ obtain the r.e.f. of } [A | 0] \text{ and write the solution in parametric vector form in terms of the free variables. The vectors multiplying each free variable give a basis for } \text{Null } A \text{ and the dimension of } \text{Null } A \text{ is the number of free variables.}
To find Col A, obtain the echelon form of $A$ (no need for r.e.f.) to determine the pivot columns of $A$. The pivot columns of $A$ (not the reduced matrix!) give a basis for Col A.

Theorem 14: Rank Thm

Rank $A := \dim \text{Col} \ A$.

$\dim \text{Nul} \ A + \dim \text{Rank} \ A = \# \text{free variables} + \# \text{basic variables} = n$ (total # of variables)

Row $A = \text{span} \ \{ \text{Rows of } A \}$

$\dim \text{Row} \ A = \dim \text{Col} \ A = \text{Rank} \ A$. 
Coordinates, Change of Basis

\[ \mathcal{B} = \{ b_1, \ldots, b_n \} \] basis for \( \mathbb{R}^n \).

For \( x \in \mathbb{R}^n \), if \( x = c_1 v_1 + \cdots + c_n v_n \), then the numbers \( c_1, \ldots, c_n \) are the coordinates of \( x \) with respect to the basis \( \mathcal{B} \).

Coordinate Mapping \( X \mapsto \left[ \begin{array}{c} c_1 \\ \vdots \\ c_n \end{array} \right] \) written as \( [x]_{\mathcal{B}} \).

If \( \mathbb{R}^n \), then the matrix of mapping

\[ P_{\mathcal{B}} = \left[ b_1, \ldots, b_n \right] \]

is the change of coordinates matrix from \( \mathcal{B} \) to the std. basis in \( \mathbb{R}^n \).

\[ X = P_{\mathcal{B}} [x]_{\mathcal{B}} \]
If we have two bases
\[ B = \{ b_1, \ldots, b_n \} \quad C = \{ c_1, \ldots, c_n \} \]
for \( V \), then the change of coordinates matrix from \( B \) to \( C \), \( \mathbf{P} \), is given by

\[
\mathbf{P} = \begin{bmatrix} [b_1]_C & \cdots & [b_n]_C \end{bmatrix}
\]

Have
\[
[x]_c = \mathbf{P} [x]_B
\]

To find \( \mathbf{P} \), either reduce
\[
[c_1 \ldots c_n | b_1 \ldots b_n] \to [1_{n \times n} | \mathbf{P}] \quad \text{or calculate } \mathbf{P}^{-1} \mathbf{P}_B \quad \text{which gives the same result.}
\]

Note
\[
\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P} \end{bmatrix}^{-1}
\]

\[ B \leftarrow C \]
Chapter 5  Eigenvalues and Eigenvectors

A \( n \times n \) matrix.

IF \( Ax = \lambda x \) has a solution for some \( x \neq 0 \) and scalar \( \lambda \), we say \( x \) is an eigenvector of \( A \) with eigenvalue \( \lambda \).

Fact.

The eigenvalues of a triangular matrix are its diagonal entries (Theorem 1, p. 206).

If \( v_1, \ldots, v_r \) are eigenvectors corresponding to distinct eigenvalues \( \lambda_1, \ldots, \lambda_r \), then \( \{v_1, \ldots, v_r\} \) is linearly independent (Theorem 2, p. 207).
Given an eigenvalue \( \lambda \), solve

\[(A - \lambda I)x = 0\]

to find the eigenspace associated with \( \lambda \).

To find the eigenvalues, solve the characteristic equation

\[\det (A - \lambda I_n) = 0\]

which is a polynomial of degree \( n \) whose roots are the eigenvalues.
Diagonalization

A is diagonal if \exists \text{ an inv. matrix } P \text{ and a diagonal matrix } D \text{ s.t.}

\[ A = PDP^{-1} \]

In this case, \( AP = PD \) and the cols. of \( P \) are (must be) e-vectors of \( A \). Say \( A \) is \underline{diagonalizable}.

Fact

\( A \) is diagonalizable \( \iff \) \( A \) has \( n \) lin. ind. e-vectors.

Not every matrix is diagonalizable!
Chapter 6  Inner Product, Length, Orthogonality

\[ u, v \in \mathbb{R}^n \text{ are orthogonal if } u \cdot v = 0 \]

Pythagorean Theorem (p. 380)  \[ u \cdot v = 0 \text{ iff } \| u + v \|^2 = \| u \|^2 + \| v \|^2 \]

Orthogonal Complements

If \( W \) is a subspace of \( \mathbb{R}^n \),

\[ W^\perp = \{ v : u \cdot w = 0 \ \forall w \in W \} \]

is another subspace of \( \mathbb{R}^n \), the orthogonal complement. We have

\[ \dim W + \dim W^\perp = n \quad (= \dim \mathbb{R}^n) \]

Theorem 3  A \( m \times n \) matrix

\( (\text{Row } A)^\perp = \text{Null } A, \quad (\text{Col } A)^\perp = \text{Null } (A^T) \)

Angles  \[ \cos \theta = \frac{u \cdot v}{\| u \| \| v \|} \]
$S = \{ u_1, \ldots, u_p \}$ is an orthogonal set if $u_i \cdot u_j = 0 \quad \forall i \neq j$

Orthogonal sets of nonzero vectors are linear independent (Thm 4, p. 384).

**Thm 5** If $\{ u_1, \ldots, u_p \}$ is an orthogonal basis for some subspace $W$ of $\mathbb{R}^n$ and $y \in W$ so that

$$y = c_1 u_1 + \cdots + c_p u_p$$

for some $c_1, \ldots, c_p$, then

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j}, \quad 1 \leq j \leq p.$$
Orthogonal Projection

W as before, y any vector in $\mathbb{R}^n$. Then

$$y = \hat{y} + z$$

where $z \in W^\perp$ and $\hat{y} \in W$ with

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

$$= (y \cdot u_1) u_1 + (y \cdot u_p) u_p$$

if we have an orthonormal basis.

$\hat{y}$ is the closest vector in $W$ to $y$,

$$\| y - \hat{y} \| \leq \| y - v \| \quad \forall v \in W.$$  

(Theorem 9, p. 398.)
Gron Schmidt Process \((P. 404)\).

Given a basis \(\{x_1, \ldots, x_p\}\) for a subspace \(W\) of \(\mathbb{R}^n\), define

\[
\begin{align*}
V_1 &= x_1, \\
V_2 &= x_2 - \frac{x_2 \cdot V_1}{V_1 \cdot V_1} V_1, \\
V_3 &= x_3 - \frac{x_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{x_3 \cdot V_2}{V_2 \cdot V_2} V_2, \\
&\quad \quad \quad \vdots \\
V_p &= x_p - \frac{x_p \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{x_p \cdot V_2}{V_2 \cdot V_2} V_2 - \cdots - \frac{x_p \cdot V_{p-1}}{V_{p-1} \cdot V_{p-1}} V_{p-1}.
\end{align*}
\]

Then \(\{v_1, \ldots, v_p\}\) is an orthogonal basis for \(W\) and

\[
\text{Span } \{v_1, \ldots, v_k\} = \text{Span } \{x_1, \ldots, x_k\}, \quad 1 \leq k \leq p.
\]