§ 9.5 Power Series

A power series about $x = a$ is a sum of constants times powers of $(x-a)$:

$$\sum_{n=0}^{\infty} C_n (x-a) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots$$

Can think of a power series as being like a polynomial of infinite degree.

Think of a as being constant and x being allowed to vary.

Each different value of x gives us a different series and the question is then which values of x give us a convergent series and which values give us a divergent series.

<u>Ex</u> $\displaystyle\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}$.

If we rewrite this as $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$, we see that each different value of x gives us a different geometric series.

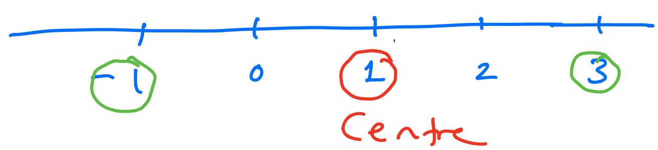
The series converges (absolutely) if the ratio $\frac{x-1}{2}$ between successive terms is less than 1 in absolute value.

ie

$$\left|\frac{x-1}{2}\right| < 1$$

$$|x-1| < 2$$

$$\Rightarrow -1 < x < 3.$$



The series diverges if

$$\left|\frac{x-1}{2}\right| \geq 1$$

$$|x-1| \geqslant 2$$

$$\Rightarrow x \leq -1 \text{ or } x \geqslant 3.$$

This example illustrates a general phenomenon about power series, namely that if x is within a certain distance either side of a, then the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges. This distance is called the <u>radius</u> <u>of</u> <u>convergence</u>, R, of the power series.

There are three possibilities which can occur.

1. The series converges only for $x=a$. In this case the radius of convergence is defined to be $R=0$.

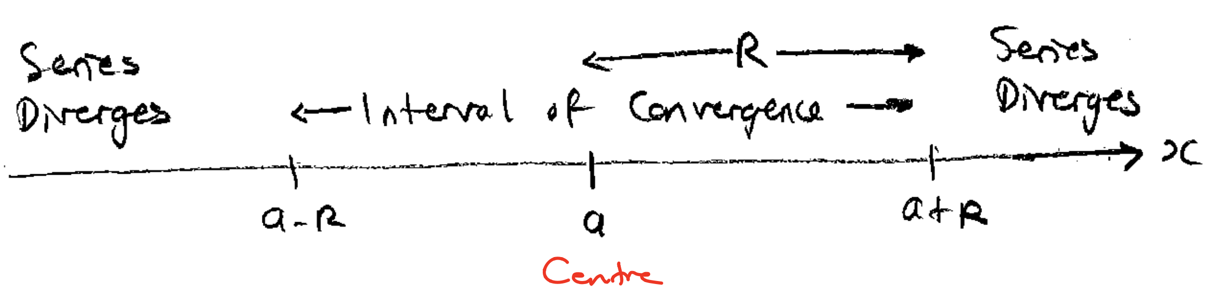
2. There exists $0<R<\infty$, called the radius of convergence, such that the series converges (absolutely) for $|x-a|<R$ and diverges for $|x-a|>R$.

3. The series converges for all values of x. In this case, the radius of convergence is defined to be $R=\infty$.

N.b. in the case $0 < R < \infty$, we know the series converges for $|x-a| < R$, ie for $a - R < x < a + R$.

We need to check the convergence of the series at the two endpts. $a-R$ & $a+R$ separately.

The interval $(a-R, a+R)$ together with those endpts. (if any) at which we have convergence together give us the <u>interval of convergence</u> of the power series.



Ex. For the previous example

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n}$$

we found that the radius of convergence was 2 while the interval of convergence was $(-1, 3)$. ← open interval, i.e. $-1 < x < 3$ endpoints NOT included!

Using the Ratio Test to Find the Radius of Convergence

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series and for each $n \geq 0$, assume $c_n \neq 0$, $x \neq a$ and set

$$a_n = c_n (x-a)^n;$$ ($\neq 0$ since $c_n \neq 0$ and $x \neq a$)

We then look at

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}$$

as we do for the ratio test.

Let's assume now that this limit does exist (where we allow the possibility that it could be ∞).

Then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|C_{n+1}(x-a)^{n+1}|}{|C_n(x-a)^n|}$$

$|ab| = |a| \cdot |b|$

$$= \lim_{n\to\infty} \frac{|C_{n+1}|\,|x-a|^{n+1}}{|C_n|\,|x-a|^n}$$

$a^b / a^c = a^{b-c}$

$$= \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} |x-a|$$

doesn't depend on n, so we can take it outside the limit.

$$= |x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$$

There are 3 possibilities.

<u>Case 1</u> $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$

In this case

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = \infty .$$

and so by the ratio test, the power series diverges for every $x \neq a$.

Note that for $x=a$, the power series is just

$$\sum_{n=0}^{\infty} C_n (a-a)^n = C_0$$

and so there is <u>always</u> convergence at $x=a$.

Hence in this case, the radius of convergence $R = 0$ $\left(= \frac{1}{\infty}\right)$

<u>Case 2</u>

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a|$$

where $K = \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$ is >0 and $<\infty$.

If we then let $R = \frac{1}{K}$, then

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a| = \frac{|x-a|}{R}.$$

By the ratio test, the series

- converges if $\frac{|x-a|}{R} < 1$ ie $|x-a| < R$
- diverges if $\frac{|x-a|}{R} > 1$ ie $|x-a| > R$.

Hence in this case the radius of convergence is R and since $R = \frac{1}{K}$ and $0 < K < \infty$, $0 < R < \infty$ also.

<u>Case 3</u> $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0.$

In this case

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-a| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|} = 0$$

and so by the ratio test, the power series converges for every x (including a).

Hence in this case, the radius of convergence $R = \infty \; \left(= \frac{1}{0}\right).$

To summarize

Method for Computing Radius of Convergence

To calculate the radius of convergence, R, for the power series $\sum_{n=0}^{\infty} C_n (x-a)^n$, use the ratio test with $a_n = C_n (x-a)^n$.

For shits and giggles

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

1) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty$, then $R=0$

2) If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = K|x-a|$ with $0 < K < \infty$,

   then $R = \frac{1}{K}$ and $0 < R < \infty$.

3). If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0$, then $R = \infty$.

Note that the ratio test tells us nothing if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ doesn't exist which can happen, for example if some of the coefficients C_n are 0.

$\underline{\text{Ex}}$. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$(0! = 1)$.

Here $C_n = \frac{1}{n!}$, so none of the C_ns are 0 and we can use the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$$

$$= |x| \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$$

$$= |x| \lim_{n\to\infty} \frac{n!}{(n+1)!}$$

$$= |x| \lim_{n\to\infty} \frac{1}{n+1}$$

$$= |x| \cdot 0$$

$$= 0.$$

This gives $R = \infty$, so the series converges for every x.

We'll see in Chapter 10 that it converges to e^x.

Ex. $\sum_{n=0}^{\infty} n!\, x^n$. $= 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots$

remember $0! = 1$

Here again all the C_n's are non-zero and we use the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$$

$$= |x| \lim_{n\to\infty} \frac{(n+1)!}{n!} = |x| \lim_{n\to\infty} \frac{(n+1)\cancel{(n)}\cancel{(n-1)}\cdots\cancel{2}\cdot\cancel{1}}{\cancel{(n)}\cancel{(n-1)}\cdots\cancel{2}\cdot\cancel{1}}$$

$$= |x| \lim_{n\to\infty} (n+1)$$

$$= \infty \quad \text{provided } x \neq 0.$$

This gives $R = 0$ $\left(= \frac{1}{\infty}\right)$ and the series converges only at 0 and diverges everywhere else.

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n+1}}{n}(x-1)^n + \cdots$

Here $C_n = \frac{(-1)^{n+1}}{n} \neq 0$ for every n, and so using the ratio test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x-1| \lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}$$

$$= |x-1| \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}}{n+1}\right|}{\left|\frac{(-1)^{n+1}}{n}\right|}$$

$$= |x-1| \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}}$$

$$= |x-1| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x-1| \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}}$$

$$= |x-1| \cdot 1.$$

Thus $K = 1$ and so $R = \frac{1}{1} = 1$. The ratio test tells us that the series converges for $|x-1| < 1$ ie $0 < x < 2$ and diverges for $|x-1| > 1$ ie $x < 0$ or $x > 2$.

Note that the ratio test tells us nothing when $|x-1| = 1$ ie when $x = 0$ or $x = 2$.

The ratio test requires $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ to exist for $a_n = C_n(x-a)^n$.

If some of the C_n's are 0, there is obviously a problem as then $a_n = 0$ for these C_n's and we can't divide by 0 when taking $\frac{|a_{n+1}|}{|a_n|}$.

One trick is to rewrite the series in such a way that all the a_n's are non-zero.

Ex. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{---} + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \text{--}$

(even powers of x 'missing')

If we write this as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1}$$

and consider it as a series in x^{2n-1} rather than in x^n, then we let

$$a_1 = x, \quad a_2 = \frac{-x^3}{3!}, \ \text{-- so} \quad a_n = \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1}.$$

Note that here $a_{n+1} = (-1)^{n+1-1} \frac{x^{2(n+1)-1}}{(2(n+1)-1)!} = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Now all the a_n's are non-zero provided $x \neq 0$ and we can use the ratio test to find that.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^n}{(2n+1)!} x^{2n+1}\right|}{\left|\frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1}\right|}$$

$$= \lim_{n\to\infty} \left| \frac{(2n-1)!}{(2n+1)!} \frac{x^{2n+1}}{x^{2n-1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{2}\cdot\cancel{1}}{(2n+1)(2n)\cancel{(2n-1)}\cancel{(2n-2)}\cdots\cancel{2}\cdot\cancel{1}} \cdot x^2 \right|$$

$$= |x^2| \lim_{n\to\infty} \frac{1}{(2n+1)(2n)}$$

$$= |x^2| \cdot 0$$

$$= 0.$$

Thus $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0 < 1$ for every x.

The ratio test then guarantees that the series converges for every x. Thus the radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$ or $\mathbb{R}$ (i.e. all real numbers).

What Happens at the Endpoints of the Interval of Convergence?

The ratio test tells us that a power series will converge inside $(a-R, a+R)$ and diverge for $x < a-R$ or $x > a+R$.

However, it tells us nothing about what happens at the endpts $a-R$, $a+R$ of the interval of convergence.

The answer depends on the series and we need to check the behaviour at the endpts separately.

Ex. $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$

We already found the radius of convergence was 1 and so the series converges for $|x-1| < 1$ ie for $0 < x < 2$.

For the endpts 0, 2 we check separately.

At $x=0$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n}$$

This is the negative of the harmonic series and so the power series diverges at $x=0$.

At $x=2$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

This is the alternating harmonic series which we already saw converges by the alternating series test.

Thus $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ converges on $(0,2]$ and diverges everywhere else.

We shall see later that this series converges to $\ln x$.

<u>Ex.</u> Find the radius and interval of convergence of the series

$$1 + 2^2x^2 + 2^4x^4 + 2^6x^6 + - + 2^{2n}x^{2n} + \cdots$$

If we simply take $a_n = 2^n x^n$ for n even and 0 for n odd, then we can't find $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$.

So instead let $a_n = 2^{2n}x^{2n}$.

Then $a_{n+1} = 2^{2(n+1)}x^{2(n+1)} = 2^{2n+2}x^{2n+2}$

and

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{2^{2n+2}x^{2n+2}}{2^{2n}x^{2n}} \right|$$

$$= \lim_{n\to\infty} |2^2x^2|$$

$$= |x^2| \lim_{n\to\infty} 4$$

$$= 4|x^2|.$$

The ratio test tells us that the series converges if $4|x^2| < 1$ ie $|x| < \frac{1}{2}$ or $-\frac{1}{2} < x < \frac{1}{2}$ and diverges if $4|x^2| > 1$ ie $|x| > \frac{1}{2}$ or $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.

At $x = \pm\frac{1}{2}$, $2^{2n} x^{2n} = 2^{2n}\left(\pm\frac{1}{2}\right)^{2n} = \frac{2^{2n}}{2^{2n}} = 1$,

so all the terms are 1 and so the series diverges by the divergence test.

Thus the radius of convergence is $\frac{1}{2}$ and the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.