

## § 9.4 Tests for Convergence

### Comparison of Series

In § 7.8 we compared two integrals to determine whether or not an improper integral converged. In the last section we compared a series with an integral to determine whether or not the series converged. Now we compare two series.

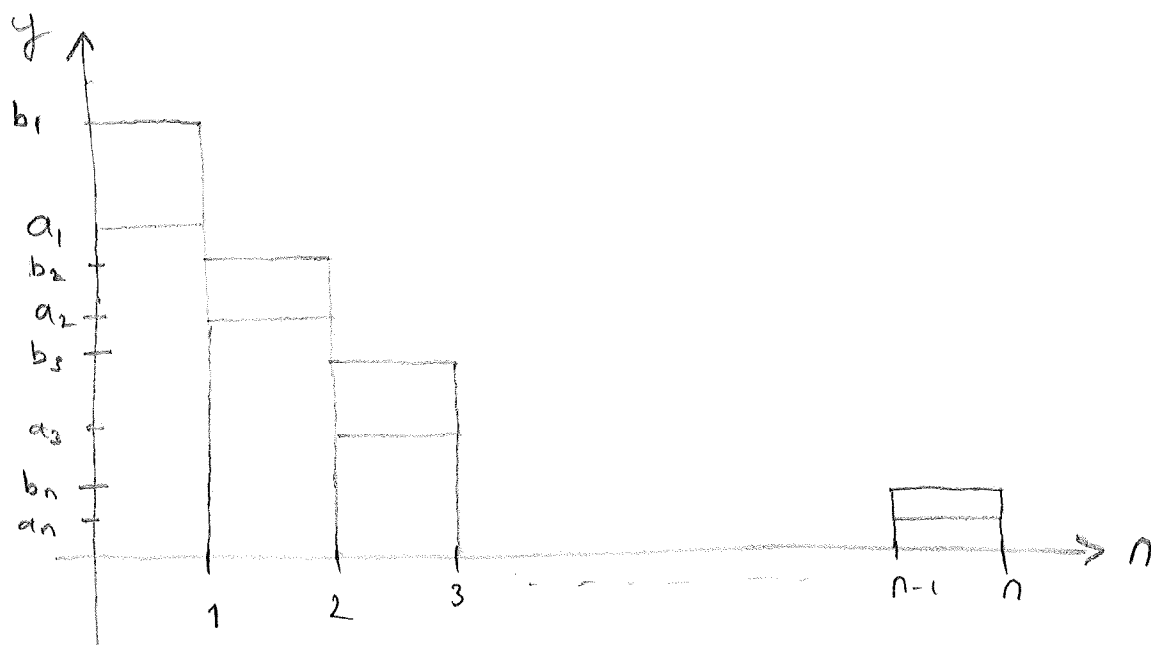
### Comparison Test

Suppose  $0 \leq a_n \leq b_n$  for every  $n$ .

i) If  $\sum b_n$  converges, then  $\sum a_n$  converges.

ii) If  $\sum a_n$  diverges, then  $\sum b_n$  diverges.

The idea behind the proof can be best understood in terms of areas using the following picture



Over each interval  $[n-1, n]$  we have a taller rectangle of height  $b_n$  and a shorter rectangle of height  $a_n$ .

Ex 
$$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$$

Since  $n^3 \leq n^3+1$  for  $n \geq 1$ ,

$$0 \leq \frac{1}{n^3} \leq \frac{1}{n^3+1} \quad \text{for } n \geq 1.$$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$  is a p-series with  $p=3 > 1$  and so

convergent. Thus  $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$  is also convergent

by the comparison test.

Ex 
$$\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$$

The convergence or divergence of this series is determined when  $n$  is large. For large  $n$ ,

$$\frac{n-1}{n^3+3} \approx \frac{n}{n^3} = \frac{1}{n^2}$$

This suggests that we should compare with  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  and as this series converges, we should expect our series to converge also.

Since a fraction increases if its numerator is made larger or its denominator is made smaller, we have

$$0 \leq \frac{n-1}{n^3+3} \leq \frac{n}{n^3+3} \leq \frac{n}{n^3} = \frac{1}{n^2}, \quad n \geq 1.$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges,  $\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$  also converges by the comparison test.

Ex. 
$$\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$$

### 1. Guess

Here, when  $n$  is large

$$\frac{6n^2+1}{2n^3-1} \approx \frac{6n^2}{2n^3} = \frac{3}{n}$$

and so we expect our series to diverge since

$$\sum_{n=1}^{\infty} \frac{3}{n} \text{ does.}$$

## 2. Comparison

Since a fraction decreases if its numerator is made smaller or its denominator is made larger, we have

$$\frac{6n^2+1}{2n^3-1} \geq \frac{6n^2}{2n^3-1} \geq \frac{6n^2}{2n^3} = \frac{3}{n}, \quad n \geq 1$$

so that

$$0 \leq \frac{3}{n} \leq \frac{6n^2+1}{2n^3-1}, \quad n \geq 1.$$

## 3. Conclusion

Thus since  $\sum_{n=1}^{\infty} \frac{3}{n}$  diverges,  $\sum_{n=1}^{\infty} \frac{6n^2+1}{3n^2-1}$  also

diverges by the comparison test.

The basic ideas behind using the comparison test can be summarized as follows.

1. Look at what happens for large  $n$  to get an idea of how the terms in the series behave.

2. If you think the series might be convergent, try to find a convergent series whose terms are (eventually) bigger than those of the series you're testing.

If you think the series might be divergent, try to find a divergent series whose terms are (eventually) smaller (and non-negative) than those of the series you're testing.

WARNING The comparison test only works for series whose terms are non-negative.

Ex.

$$\sum_{n=3}^{\infty} \frac{n^2 - 5}{n^3 + n + 2}$$

As  $n \rightarrow \infty$ ,  $n^2$  dominates in the numerator and  $n^3$  in the denominator so that

$$\frac{n^2 - 5}{n^3 + n + 2} \approx \frac{n^2}{n^3} = \frac{1}{n} \quad \text{for } n \text{ large.}$$

So  $\frac{n^2 - 5}{n^3 + n + 2}$  behaves like  $\frac{1}{n}$  and

since  $\sum_{n=3}^{\infty} \frac{1}{n}$  diverges, we would expect that

$$\sum_{n=3}^{\infty} \frac{n^2 - 5}{n^3 + n + 2} \quad \text{to diverge also.}$$

However, for  $n \geq 3$ ,

$$0 \leq \frac{n^2 - 5}{n^3 + n + 2} < \frac{n^2}{n^3} = \frac{1}{n}$$

and the inequality goes in the wrong direction for using the comparison test.

Using algebra, we could find  $k > 0$  such that for  $n$  large enough

$$\frac{n^2 - 5}{n^3 + n + 2} \geq \frac{k}{n} \geq 0$$

and since  $\sum \frac{k}{n}$  diverges, we could use the comparison test in the usual way to conclude that  $\sum_{n=3}^{\infty} \frac{n^2 - 5}{n^3 + n + 2}$  diverges.

However, there is an easier way-----

### Limit Comparison Test

Suppose  $a_n > 0$  and  $b_n > 0$  for all  $n$ . If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \quad \text{where } c > 0,$$

then the two series  $\sum a_n$ ,  $\sum b_n$  either both converge or both diverge.



e.g. in the previous example if we let

$$a_n = \frac{n^2 - 5}{n^3 + n + 2}$$

and  $b_n = \frac{1}{n}$ , then for  $n \geq 3$ .

$$\frac{a_n}{b_n} = \frac{\frac{n^2 - 5}{n^3 + n + 2}}{\frac{1}{n}} = \frac{n^3 - 5n}{n^3 + n + 2}$$

and if we divide above and below by  $n^3$  we get

$$\frac{a_n}{b_n} = \frac{1 - \frac{5}{n^2}}{1 + \frac{1}{n^2} + \frac{2}{n^3}} \longrightarrow \frac{1}{1} = 1 \quad \text{as } n \rightarrow \infty.$$

Thus  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1 > 0$  and since

$\sum_{n=3}^{\infty} \frac{1}{n}$  diverges,  $\sum_{n=3}^{\infty} \frac{n^2 - 5}{n^3 + n + 2}$  also diverges.

Ex  $\sum \frac{n^2 + 6}{n^4 - 2n + 3}$

For  $n$  large

$$a_n = \frac{n^2 + 6}{n^4 - 2n + 3} \approx \frac{n^2}{n^4} = \frac{1}{n^2}$$

and so we take  $b_n = \frac{1}{n^2}$ .

Then

$$\frac{a_n}{b_n} = \frac{\frac{n^2 + 6}{n^4 - 2n + 3}}{\frac{1}{n^2}} = \frac{n^4 + 6n^2}{n^4 - 2n + 3}$$

$$= \frac{1 + \frac{6}{n^2}}{1 - \frac{2}{n^3} + \frac{3}{n^4}}$$

(divide above and below by  $n^4$ )

$$\rightarrow \frac{1}{1} = 1 \text{ as } n \rightarrow \infty.$$

Limit comparison test then applies with  $c = 1$ .

Since  $\sum \frac{1}{n^2}$  converges, by the limit comparison test

$\sum \frac{n^2 + 6}{n^4 - 2n + 3}$  also converges.

Ex  $\sum \sin\left(\frac{1}{n}\right)$

Since  $\sin x \approx x$  for  $x$  small, this suggests we try the limit comparison test with  $a_n = \sin\left(\frac{1}{n}\right)$  and  $b_n = \frac{1}{n}$ .

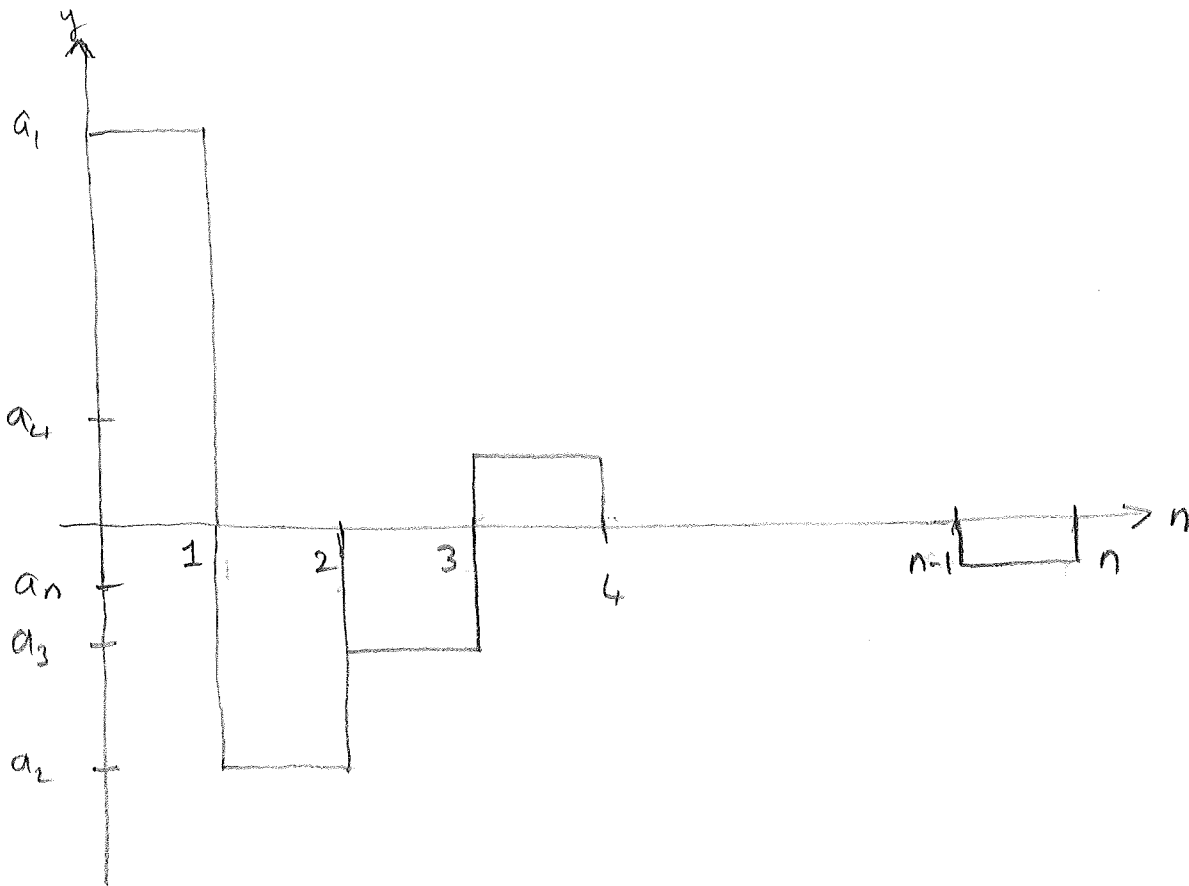
Then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Thus  $c=1$  and since  $\sum \frac{1}{n}$  diverges,  $\sum \sin\left(\frac{1}{n}\right)$  also diverges.

## Series with both Positive and Negative Terms

If  $\sum a_n$  has both positive and negative terms, then we cannot interpret  $\sum a_n$  in terms of area in the same way as before



Note that the area of each rectangle is  $|a_n|$ , so we can still ask if the sum of the (unsigned) areas  $\sum |a_n|$  is finite.

This leads to the following definition.

We say the series  $\sum a_n$  is absolutely convergent if the series

$$\sum |a_n|$$

is convergent.

Fact Absolute Convergence Implies Regular Convergence

If  $\sum |a_n|$  converges, then  $\sum a_n$  converges.

Ex. 
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$

Here  $a_n = \frac{(-1)^n}{n^3}$  and since  $|a_n| = \frac{1}{n^3}$  and

$\sum_{n=1}^{\infty} \frac{1}{n^3}$  is a convergent  $p$ -series ( $p=3 > 1$ ),

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$  converges also.

# Comparison with a Geometric Series

## - the Ratio Test

For a geometric series

$$\sum a_n = \sum ax^n$$

the ratio between terms is

$$\frac{a_{n+1}}{a_n} = \frac{ax^{n+1}}{ax^n} = x$$

which is constant.

Recall that if  $|x| < 1$ , then the series converges and if  $|x| \geq 1$ , the series diverges.

For many other series  $\frac{a_{n+1}}{a_n}$  may not be constant, but we can still say something about convergence.

## The Ratio Test

For a series  $\sum a_n$  with  $a_n \neq 0$  for  $n$  sufficiently large, suppose that the sequence of ratios  $\frac{|a_{n+1}|}{|a_n|}$  has a limit

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L.$$

- i) If  $L < 1$ , then  $\sum a_n$  converges absolutely and is thus convergent.
- ii) If  $L > 1$  or  $L$  is infinite (i.e.  $\frac{|a_{n+1}|}{|a_n|}$  grows without limit), then  $\sum a_n$  diverges.
- iii) If  $L = 1$ , the test tells us nothing about the convergence or divergence of  $\sum a_n$ .

## Idea of Proof

Convergent case  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L < 1.$

Let  $x$  be a number between  $L$  and  $1$ ,  
i.e.  $L < x < 1.$

Then for all  $n$  suff. large, say  $n \geq k$ , we have

$$\frac{|a_{n+1}|}{|a_n|} < x.$$

(  $\frac{|a_{n+1}|}{|a_n|}$  is getting close to  $L$ , so eventually it must come below  $x$  and stay below  $x$  ).

Then  $|a_{k+1}| < |a_k| x$

$$|a_{k+2}| < |a_{k+1}| x < |a_k| x \cdot x = |a_k| x^2$$

$$|a_{k+3}| < |a_{k+2}| x < |a_k| x^2 \cdot x = |a_k| x^3$$

⋮

etc.



The general pattern we see here is that for  $i \geq 0$ ,

$$|a_{k+i}| \leq |a_k| x^i.$$

Comparison with the convergent geometric series

$$\sum_{n=k}^{\infty} |a_n| x^k$$

shows that

$$\sum_{n=k}^{\infty} |a_n|$$

is also convergent. Since the first few terms of a series do not affect the convergence or divergence (§9.3 Property 2), it follows that

$$\sum |a_n|$$

is convergent. Thus  $\sum a_n$  is absolutely convergent and hence convergent.

Divergent Case  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L > 1 \text{ or } \infty.$

In this case, if we pick  $x$  such that

$$L > x > 1,$$

then for sufficiently large  $n$ , say  $n > m$

$$|a_{n+1}| \geq |a_n| x > |a_n|.$$

Thus the sequence  $|a_n|$  is (eventually) increasing and so the terms  $a_n$  cannot converge to 0. By the divergence test

$\sum a_n$  must diverge.

Ex  $\sum_{n=1}^{\infty} \frac{1}{n!}$

Here  $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$ , the product of the first  $n$  natural numbers.

Here  $a_n = \frac{1}{n!}$  and

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!}$$

$$= \frac{n(n-1) \dots 2 \cdot 1}{(n+1)(n)(n-1) \dots 2 \cdot 1}$$

$$= \frac{1}{n+1} \quad (\text{all other terms cancel!})$$

Thus  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$

and so  $\sum_{n=1}^{\infty} \frac{1}{n!}$  converges by the ratio test.

Ex.

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$$

Here

$$a_n = \frac{(2n)!}{n!(n+1)!}$$

so

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(2n+2)!}{(n+1)!(n+2)!}}{\frac{(2n)!}{n!(n+1)!}}$$

$$= \frac{(2n+2)! n!(n+1)!}{(2n)! (n+1)! (n+2)!}$$

$$= \frac{(2n+2)! n!}{(2n)! (n+2)!}$$

$$= \frac{[(2n+2)(2n+1)(2n) \dots 2 \cdot 1][n(n-1) \dots 2 \cdot 1]}{[(2n)(2n-1) \dots 2 \cdot 1][(n+2)(n+1)(n) \dots 2 \cdot 1]}$$

$$= \frac{(2n+2)(2n+1)}{(n+2)(n+1)}$$

Now divide every term above and below by  $n$  (rather like in the limit comparison test) to get

$$\frac{(2 + \frac{2}{n})(2 + \frac{1}{n})}{(1 + \frac{2}{n})(1 + \frac{1}{n})} \rightarrow \frac{2 \cdot 2}{1 \cdot 1} = 4 \text{ as } n \rightarrow \infty.$$

Thus

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = 4 > 1 \quad \text{and so} \quad \sum_{n=1}^{\infty} \frac{(2n)!}{n!(n+1)!}$$

is divergent.

Ex. 
$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n(3)^n}$$

Here  $a_n = \frac{(-2)^n}{n(3)^n}$  and

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left| \frac{(-2)^{n+1}}{(n+1)(3)^{n+1}} \right|}{\left| \frac{(-2)^n}{n(3)^n} \right|}$$

$$= \frac{\frac{2^{n+1}}{(n+1)(3)^{n+1}}}{\frac{2^n}{n(3)^n}}$$

$$= \frac{2^{n+1} \cdot n(3)^n}{2^n \cdot (n+1)(3)^{n+1}}$$

$$= \frac{2n}{(n+1) \cdot 3}$$

$$= \frac{2}{3} \cdot \frac{n}{n+1}$$

$$= \frac{2}{3} \cdot \frac{1}{1 + \frac{1}{n}} \quad (\text{divide above and below by } n)$$

$$\rightarrow \frac{2}{3} \quad \text{as } n \rightarrow \infty.$$

Thus  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \frac{2}{3} < 1$  and so the

series  $\sum_{n=1}^{\infty} \frac{(-2)^n}{n(3)^n}$  converges (absolutely) by

the ratio test.

Ex. a).  $\sum_{n=1}^{\infty} \frac{1}{n}$  - harmonic series.

$$\text{Here } \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

So  $L = 1$  in this case and we know this series is divergent from the last section.

b)  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  - p series ( $p=2$ ).

$$\text{Here } \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

So  $L = 1$  in this case also and we know this series is convergent ( $p > 1$ ) from the last section.

Taken together, these two examples show why the ratio test tells us nothing when  $L = 1$ .

# Alternating Series

These are series where the terms alternate in sign e.g.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots + \frac{(-1)^{n-1}}{n} + \dots$$

For these series we have the

## Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots + (-1)^{n-1} a_n + \dots$$

converges if

$0 < a_{n+1} < a_n$  for all  $n$  and

where it's at!

$$\lim_{n \rightarrow \infty} a_n = 0.$$

← note that we need this condition for any convergent series.



Ex The previous series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$\frac{1}{n} > \frac{1}{n+1} > 0$$

and  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$

is easily seen to satisfy the convergence criteria for the alternating series test and is thus convergent.

Note that this series is not absolutely convergent as

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \quad (\text{harmonic series})$$

which we know is divergent.

A series such as  $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$  which is convergent but not absolutely convergent is known as conditionally convergent.

# Behaviour of the Partial Sums of an Alternating Series.

Let  $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$  be an alternating series where  $0 < a_{n+1} < a_n$  for every  $n$  and  $\lim_{n \rightarrow \infty} a_n = 0$ .

Let  $S_n = \sum_{i=1}^n (-1)^{i-1} a_i$  be the  $n$ -th partial sum.

Then  $S_1 = a_1 > 0$

$S_2 = a_1 - a_2 > 0$  as  $a_2 < a_1$ , but clearly also

$S_2 < a_1$  as  $a_2 > 0$ .

$S_3 = a_1 - a_2 + a_3 > a_1 - a_2 = S_2$  but

$a_1 - a_2 + a_3 = a_1 - (a_2 - a_3) < a_1 = S_1$

as  $a_2 - a_3 > 0$  since  $a_2 > a_3$ .

$$S_0 \quad S_1 > 0, \quad 0 < S_2 < S_1,$$

$$S_2 < S_3 < S_1, \quad \text{and this pattern}$$

$$\text{continues e.g. } S_2 < S_4 < S_3$$

$$S_4 < S_5 < S_3$$

etc.

Since  $|S_n - S_{n-1}| = |a_n| \rightarrow 0$  as  $n \rightarrow \infty$ ,

the partial sums oscillate inwards  
to a fixed limit  $S$



This is the basic idea behind the proof for  
our criterion for the convergence of alternating  
series.

We can also see that

$$S_2 < S_4 < S_6 < \dots < S < \dots < S_3 < S_1$$

Thus, for any  $n$  (even or odd)

$$|S - S_n| < |S_{n+1} - S_n| = |a_{n+1}|.$$

This allows us to estimate the error involved in summing only finitely many terms of an alternating series.

Ex. Estimate the error in approximating the sum of the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad (= \ln 2 \approx 0.693)$$

by taking the sum of the first 9 terms.

$$S_9 = 1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{1}{9} = 0.7456\dots$$

The error in using  $S_9$  as an approx. for  $S$  is then bounded by

$$|a_{10}| = \left| -\frac{1}{10} \right| = 0.1.$$

## Recognizing Series - Which Test to Use

With so many tests available, one needs to know which ones to apply to a given series to check for convergence or divergence.

I suggest the following rough guidelines.

# Steps for Testing a Series $\sum a_n$

1. Check  $\lim_{n \rightarrow \infty} a_n$ . If this isn't 0 or doesn't exist, then  $\sum a_n$  is divergent by the divergence test.

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n+1}$$

alternating, but not convergent

2. Is the series alternating? If yes, then try the alternating series test.

3. If  $a_n$  contains factorials or powers involving  $n$ , e.g.  $(2n)!$ ,  $2^n$ ,  $e^{-n^2}$ , etc, then try the ratio test.

4. If  $a_n$  contains only fixed powers of  $n$ , e.g.  $n^2$ ,  $(n-1)^3$ ,  $\sqrt{2n^2+6}$  etc., then try the comparison test or the limit comparison test.

N.b. you need to have non-negative terms for the regular comparison test and positive terms for the limit comparison test.

# DISCLAIMER

This system isn't foolproof and you need to be flexible.

As with methods of integration, if one test doesn't work, you need to try another.