§ 9.2, 9.3 Geometric Series

Convergence of Series

A series is a sum of terms. Series can be either

- finite: $\sum_{n=1}^{N} a_n := a_1 + a_2 + \cdots + a_N$
- infinite: $\sum_{n=1}^{\infty} a_n := a_1 + a_2 + \cdots + a_n + \cdots$

Examples

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n \quad \left(= \frac{n(n+1)}{2}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \quad \left(= \frac{\pi^2}{6}!\right)$$

A geometric series is a series (finite or infinite) where the ratio between successive terms is fixed.

e.g.

$$\sum_{n=0}^{5} 3(2)^n = \underset{\overset{\|}{3\cdot 2^0 = 3\cdot 1 = 3}}{3} + \underset{\overset{\|}{3\cdot 2^1 = 3\cdot 2 = 6}}{6} + 12 + 24 + 48 + 96$$

$$\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n = 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \frac{1}{729} - \cdots$$

In general

a finite geometric series has the form

$$\sum_{i=0}^{n-1} ax^i = \underbrace{a + ax + ax^2 + \cdots + ax^{n-1}}_{n \text{ terms}}$$

while an infinite geometric series has the form

$$\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + \cdots + ax^{n-1} + ax^n + \cdots$$

Summing a Finite Geometric Series

Let S_n denote the sum

$$S_n = a + ax + \cdots + ax^{n-1} = \sum_{i=0}^{n-1} ax^i \quad (n \text{ terms}).$$

Then

$$xS_n = ax + ax^2 + \cdots + ax^n$$

And so

$$S_n - xS_n = a + \cancel{ax} + \cancel{} \cdots + \cancel{ax^{n-1}}$$
$$- \cancel{ax} - \cancel{ax^2} \cdots - \cancel{ax^{n-1}} - ax^n$$

$$= a - ax^n \quad \text{(all terms cancel except the first and the last)}$$

$$= a(1 - x^n).$$

So

$$S_n - x S_n = a(1-x^n)$$

$$S_n(1-x) = a(1-x^n)$$

and if we divide by $1-x$ (which we can do provided $x \neq 1$), we get

$$S_n = a + ax + \cdots + ax^{n-1} = a \cdot \frac{1-x^n}{1-x}, \quad x \neq 1.$$

Note that if $x = 1$, then

$$S_n = \underbrace{a + a + \cdots + a}_{n \text{ terms}} = na.$$

Summarize

$$a + ax + \cdots\cdots + ax^{n-1} = \begin{cases} a\frac{1-x^n}{1-x}, & x \neq 1 \\ na, & x = 1 \end{cases}$$

Infinite Geometric Series

Recall that the sequence $\{x^n\}_{n=1}^{\infty}$ was convergent to 0 if $|x| < 1$ and divergent if $|x| > 1$.

Thus if $|x| < 1$

$$S_n = a\frac{(1-\underbrace{x^n}_{\to 0 \text{ as } n\to\infty})}{1-x} \longrightarrow \frac{a}{1-x} \quad \text{as } n \longrightarrow \infty.$$

In this case we say that the infinite geometric series

$$\sum_{n=0}^{\infty} a x^n$$

converges and has sum

$$S = a + ax + ax^2 + \cdots + ax^n + \cdots = \frac{a}{1-x}$$

which we can also write as

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}, \qquad |x| < 1.$$

If $|x| > 1$ then provided $a \neq 0$,

$|ax^n| \longrightarrow \infty$ as $n \to \infty$ and so

S_n cannot converge. $\quad a\dfrac{1-x^n}{1-x} \longrightarrow |x^n| \to \infty$ as $n \to \infty$

If $x = 1$, then provided $a \neq 0$,

$S_n = a + \cdots + a = na$

which also doesn't converge.

Finally, if $x = -1$, then provided $a \neq 0$

$$S_n = \underbrace{(a - a)}_{=0} + \underbrace{(a - a)}_{=0} + \cdots + (-1)^{n-1}a = \begin{cases} a, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

which also doesn't converge.

Thus, if $|x| \geq 1$, provided $a \neq 0$, the sequence S_n diverges and we say that the infinite series

$$\sum_{n=0}^{\infty} ax^n \quad \text{diverges.}$$

Ex a) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n$

Here $a=1$, $x=\frac{1}{2}$ and $\left|\frac{1}{2}\right|<1$, so the series is convergent with sum

$$\frac{\overset{a}{1}}{\underset{1-x}{1-\frac{1}{2}}} = 2.$$

$$\frac{2^{nd}}{1st} = \frac{ax}{a} = x$$

b) $3-1+\frac{1}{3}-\frac{1}{9}+\frac{1}{27}-\cdots = \sum_{n=0}^{\infty} 3\left(-\frac{1}{3}\right)^n$

the first term always gives you a.
dividing the second term by the first gives you x, in this case $x=-1/3$.

Here $a=3$, $x=-\frac{1}{3}$ and $\left|-\frac{1}{3}\right|<1$, so the series is convergent with sum

$$\frac{3}{1-\left(-\frac{1}{3}\right)} = \frac{3}{4/3} = \frac{9}{4}.$$

c) $1-7+49-343+\cdots = \sum_{n=0}^{\infty}(-7)^n$

Here $a=1$, $x=-7$ and $|-7|\geq 1$, so the series is divergent.

The situation with geometric series is an example of the more general phenomenon of convergence or divergence of a general infinite series.

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series and for each $n \geq 1$ define the nth partial sum

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

The numbers

$$S_1, S_2, S_3, \ldots, S_n, \ldots$$

give us an infinite sequence, the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$.

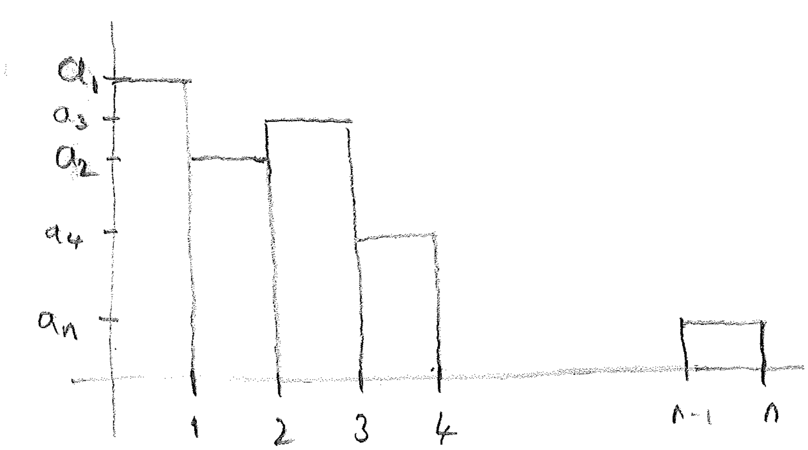
If $\{S_n\}_{n=1}^{\infty}$ is convergent with some (finite) limit S, then we say that $\sum_{n=1}^{\infty} a_n$ is convergent with sum S and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

Otherwise if $\{S_n\}_{n=1}^{\infty}$ is divergent, we say $\sum_{n=1}^{\infty} a_n$ is divergent.

Visualizing Series

If we make the following graph where each rectangle over the interval $[n-1, n]$ has height a_n, then $\sum_{n=1}^{\infty} a_n$ represents the sum of all the areas of the rectangles. This is basically an improper integral of type $\int_a^{\infty} f(x)\,dx$.



Convergence Properties of Series

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge and k is a constant, then

   i) $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

   ii) $\sum_{n=1}^{\infty} k a_n$ converges to $k \sum_{n=1}^{\infty} a_n$.

2. Changing a finite number of terms in a series does not change whether or not it converges, although it may change the value of its sum if it does converge.

3. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

   Equivalently if $\lim_{n\to\infty} a_n \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. (Divergence test)

4. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} k a_n$ diverges if $k \neq 0$.

Ex. $\displaystyle\sum_{n=1}^{\infty} (1-e^{-n})$

$$1-e^{-n} \longrightarrow 1-0 = 1 \quad \text{as} \quad n \to \infty.$$

Since the limit of the terms is not 0, this series must diverge by Property 3 above.

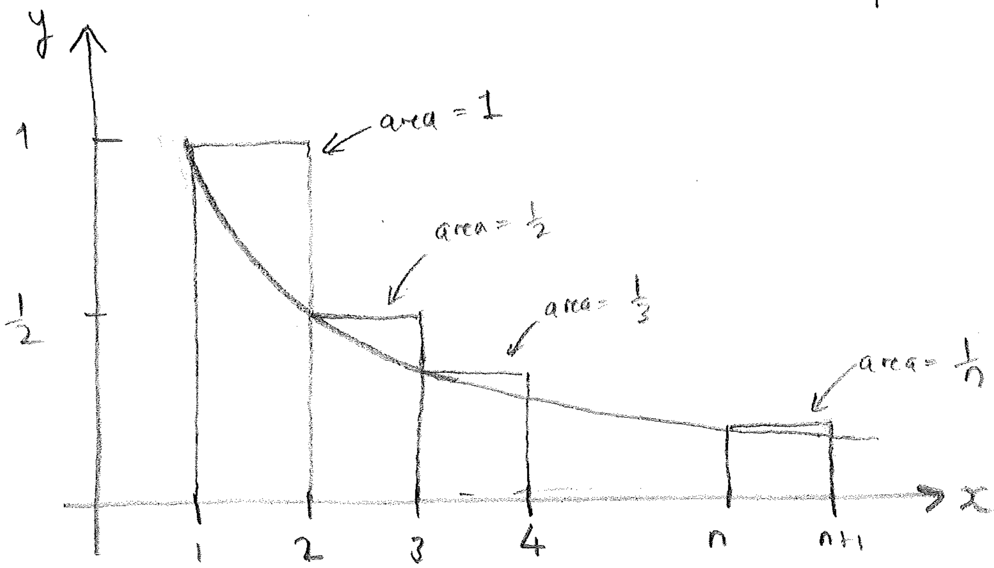
Ex. The Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

Claim this series diverges.

See this by approximating $\int_1^{\infty} \frac{1}{x}\,dx$ as a left-hand sum.

(Recall that $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.)



Since $\frac{1}{x}$ is decreasing, we see by looking at areas that

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \int_1^{n+1} \frac{1}{x}\,dx = \left[\ln x\right]_1^{n+1}$$

$$= \ln(n+1) - 0.$$

Since $\ln(n+1) \to \infty$ as $n \to \infty$, $S_n \to \infty$ as $n \to \infty$ and so $\sum_{n=1}^{\infty} \frac{1}{n}$ does indeed diverge.

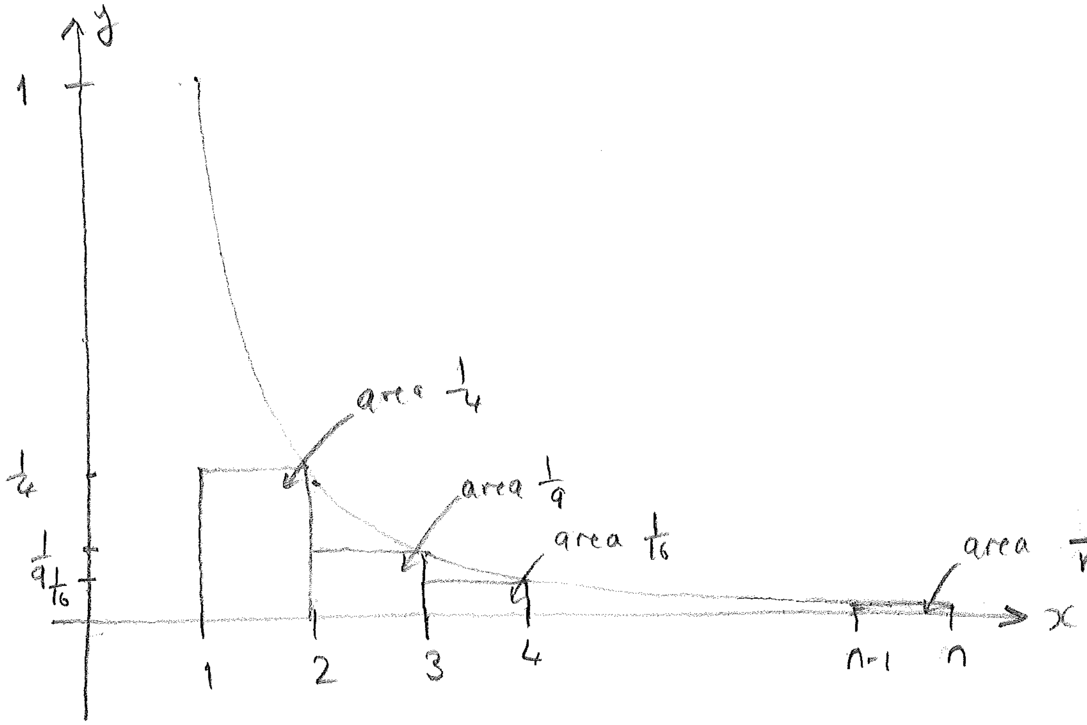
So $a_n \to 0$ as $n \to \infty$ does not necessarily mean $\sum_{n=1}^{\infty} a_n$ converges.

Remark There is also a more elementary way of doing this which doesn't use integration.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

In this case, we compare with $\int_1^{\infty} \frac{1}{x^2}\,dx$ and since this integral converges, we guess that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ should also converge.

In this case we should use a right-hand sum.



Again by area we see

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq \int_1^n \frac{1}{x^2}\,dx$$

$$= \left[-\frac{1}{x}\right]_1^n$$

$$= -\frac{1}{n} - (-1).$$

Thus

$$\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} \leq 1 - \frac{1}{n}$$

and so, adding 1 to both sides

$$S_n = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

$$< 2.$$

The sequence $\{S_n\}_{n=1}^{\infty}$ is increasing as we're always adding positive terms $\frac{1}{n^2}$ and we've just shown that is bounded above and hence bounded.

By our earlier results on sequences, we can then say that $\{S_n\}_{n=1}^{\infty}$ converges and hence so does

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

In fact, Euler showed that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

<u>Remark</u> One can also show $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, but what about $\sum_{n=1}^{\infty} \frac{1}{n^3}$? If you can do this one, you'll probably get a Fields medal!

The method of the last two examples can be used to prove the following:

The Integral Test

Suppose $a_n = f(n)$, where $f(x)$ is continuous, decreasing, and positive for $x \geq c$.

i) If $\int_c^\infty f(x)\,dx$ converges, then $\sum a_n$ converges.

ii) If $\int_c^\infty f(x)\,dx$ diverges, then $\sum a_n$ diverges.

Recall that we showed in § 7.7 that

$$\int_1^\infty \frac{1}{x^p}\,dx$$

was convergent for $p>1$ and divergent for $p \leq 1$.

Now let us look at the series $\sum_{n=1}^\infty \frac{1}{n^p}$ (p-series).

First, if $p \leq 0$, $\frac{1}{n^p} = n^{-p}$ does not tend to 0 as $n \to \infty$. Hence in this case

$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ diverges by the divergence test (Property 3)}$$

On the other hand, if $p>0$, then $\frac{1}{x^p}$ is a positive decreasing function for $x \geq 1$ and we can apply the integral test.

Hence, in this case, we can say that

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $0 < p \leq 1$ and converges for $p > 1$.

We can summarize what we have found as follows:

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and

diverges if $p \leq 1$.