Part II Trigonometric Substitutions

Substitutions of $\sin\theta$ or $\tan\theta$ which are useful for integrands involving square roots of quadratics or unfacterable quadratics.

Sine Substitutions

Make use of $\cos^2\theta + \sin^2\theta = 1$ to simplify integrands involving $\sqrt{a^2-x^2}$. by letting $x = a\sin\theta$

Ex.

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$\sqrt{a^2-x^2}$ suggests we let

$$x = a\sin\theta, \quad dx = a\cos\theta\,d\theta$$

$$\frac{x}{a} = \sin\theta, \quad \theta = \arcsin\left(\frac{x}{a}\right)$$

Then

$$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$$

$$= \sqrt{a^2(1-\sin^2\theta)}$$

$$= a\sqrt{1-\sin^2\theta}$$

$$= a\sqrt{\cos^2\theta}$$

$$= a\cos\theta$$

Rewrite the integral in terms of θ

dx

$$\int \frac{1}{\sqrt{a-x^2}} = \int \frac{a\cos\theta \; d\theta}{a\cos\theta}$$

$\sqrt{a^2-x^2}$

$$= \int d\theta$$

$$= \theta + C$$

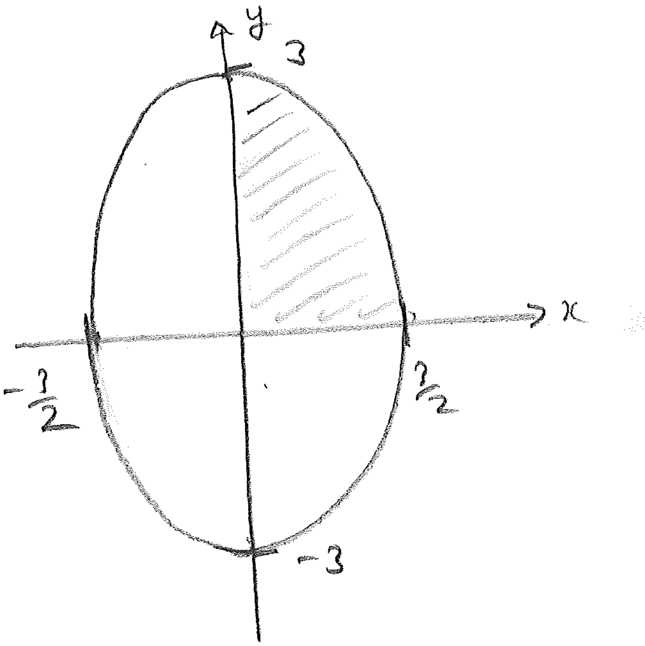
Convert back to x

$$= \arcsin\left(\frac{x}{a}\right) + C$$

Ex. Find the area of the ellipse

$$4x^2 + y^2 = 9.$$

Ellipse looks like



$$y^2 = 9 - 4x^2$$

$$y = \pm\sqrt{9-4x^2}$$

$y = \sqrt{9-4x^2}$ gives the upper half of the ellipse.

From the picture, by symmetry

$$A = 4\int_0^{\frac{3}{2}} \sqrt{9-4x^2}\,dx = 4\,(\text{area of top right quarter})$$

$$\sqrt{9-4x^2} = \sqrt{3^2-(2x)^2},$$

So we want

$$2x = 3\sin\theta$$

$$x = \frac{3}{2}\sin\theta$$

Then $\theta = \arcsin\left(\frac{2x}{3}\right)$ and

$dx = \frac{3}{2}\cos\theta \, d\theta$ while.

$$\sqrt{9-4x^2} = \sqrt{9-4\cdot\frac{9}{4}\sin^2\theta}$$

$$= \sqrt{9-9\sin^2\theta}$$

$$= 3\sqrt{1-\sin^2\theta}$$

$$= 3\sqrt{\cos^2\theta}$$

$$= 3\cos\theta$$

<u>Limits</u> When $x=0$, $\theta = 0$

$x = \frac{3}{2}$, $\theta = \frac{\pi}{2}$.

So

$$A = 4 \int_0^{\frac{\pi}{2}} 3\cos\theta \cdot \frac{3}{2}\cos\theta \, d\theta$$

$$= 18 \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

Can do this by the table (IV-18)

or we can use the identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

to get

$$A = 18 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= 9 \int_0^{\frac{\pi}{2}} d\theta + 9 \int_0^{\frac{\pi}{2}} \cos 2\theta \, d\theta$$

$$= 9 [\theta]_0^{\frac{\pi}{2}} + 9 \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

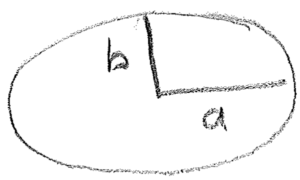
$$= 9\left(\frac{\pi}{2} - 0\right) + 9(0 - 0) = \frac{9\pi}{2}$$

<u>Notes</u>. For integrals of the type

$\int \sin^2\theta \, d\theta$, use the identity

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

The area of an ellipse



as shown is πab.

In our case $a = 3$, $b = \frac{3}{2}$

and $A = \pi(3)(\frac{3}{2}) = \frac{9\pi}{2}$.

Of course, if $a = b$, we get

$$A = \pi a^2$$

as we'd expect!

Tangent Substitutions

Integrals involving a^2+x^2 can be simplified by letting $x = a\tan\theta$
and using the identity $\sec^2\theta = 1 + \tan^2\theta$

[Book says using $\sin^2\theta + \cos^2\theta = 1$
and $\tan\theta = \frac{\sin\theta}{\cos\theta}$ which is basically the same]

Ex

$$\int \frac{1}{a^2+x^2}\,dx$$

Let $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$

$\frac{x}{a} = \tan\theta$

$\theta = \arctan\left(\frac{x}{a}\right)$

$$a^2+x^2 = a^2 + a^2\tan^2\theta$$
$$= a^2(1+\tan^2\theta)$$

$$= a\sec^2\theta \qquad \text{as} \qquad \sec^2\theta = 1 + \tan^2\theta.$$

Rewrite

$$\int \frac{1}{a^2+x^2}\,dx = \int \frac{\overbrace{a\sec^2\theta\ d\theta}^{dx}}{\underbrace{a^2\sec^2\theta}_{a^2+x^2}}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{\theta}{a} + C$$

$$= \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

Ex. Use a tangent substitution to show

$$\int_0^1 \sqrt{1+x^2}\, dx = \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$$

and interpret these integrals in terms of area.

Let $x = \tan\theta$

$\theta = \arctan x$

$dx = \sec^2\theta \, d\theta$

$$\sqrt{1+x^2} = \sqrt{1+\tan^2\theta}$$

$$= \sqrt{\sec^2\theta}$$

$$= \sec\theta$$

Limits

When $x = 0$, $\theta = 0$

$x = 1$, $\theta = \frac{\pi}{4}$ $\quad$ ($\tan\frac{\pi}{4} = 1$).

Rewrite.

$$\int_0^1 \sqrt{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \overbrace{\sec\theta}^{\sqrt{1+x^2}} \cdot \underbrace{\sec^2\theta\, d\theta}_{dx}$$

$$= \int_0^{\frac{\pi}{4}} \sec^3\theta\, d\theta$$

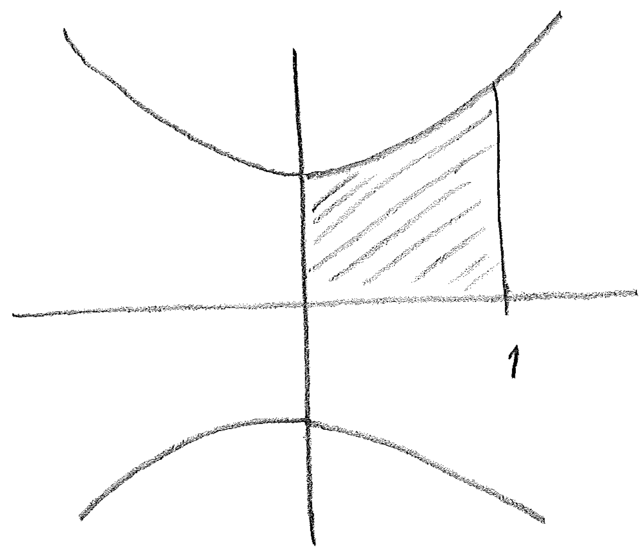
$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\theta}\, d\theta.$$

<u>Interpretation</u>

Let $y = \sqrt{1+x^2}$

$y^2 = 1+x^2$

$y^2 - x^2 = 1$



This is a hyperbola and the integral is the area under the part of the hyperbola as shown.

Completing the Square to Use a Trigonometric Substitution.

Ex. $$\int \frac{3}{\sqrt{2x-x^2}}$$

Need to complete the square in $2x-x^2$

$$2x-x^2 = -(x^2-2x)$$

$$= -(x^2-2(1)x)$$

$$= -(x^2-2(1)x+1^2-1^2)$$

$$= -((x-1)^2-1^2)$$

$$= 1-(x-1)^2.$$

Rewrite the integral

$$\int \frac{3}{\sqrt{2x-x^2}}\,dx = \int \frac{3}{\sqrt{1-(x-1)^2}}\,dx$$

Suggests the subst.

$$x-1 = \sin\theta$$

$$\theta = \arcsin(x-1).$$

$$dx = \cos\theta\, d\theta$$

So

$$\int \frac{3}{\sqrt{1-(x-1)^2}}\,dx = \int \frac{3\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$$

$$= \int \frac{3\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta$$

$$= \int \frac{3\cos\theta}{\cos\theta}\,d\theta$$

$$= 3\int d\theta$$

$$= 3\theta + C$$

$$= 3\arcsin(x-1) + C$$

Ex. $\displaystyle\int \frac{1}{x^2+x+1}\,dx$

Completing the square, we get

$$x^2+x+1 = x^2+2\left(\tfrac{1}{2}\right)x+1$$

$$= x^2+2\left(\tfrac{1}{2}\right)x+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+1$$

$$= \left(x+\tfrac{1}{2}\right)^2+\frac{3}{4} = \left(x+\tfrac{1}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}\right)^2$$

So $\displaystyle\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

Suggests the tangent substitution

$$x+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$$

$$\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) = \tan\theta$$

$$\theta = \arctan\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

$$dx = \frac{\sqrt{3}}{2} \sec^2\theta \, d\theta$$

$$\left(x+\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2 = \left(\tfrac{\sqrt{3}}{2}\right)^2 \tan^2\theta + \left(\tfrac{\sqrt{3}}{2}\right)^2$$

$$= \left(\tfrac{\sqrt{3}}{2}\right)^2 (\tan^2\theta + 1)$$

$$= \left(\tfrac{\sqrt{3}}{2}\right)^2 (\sec^2\theta)$$

$$\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\left(\frac{\sqrt{3}}{2}\right)^2 \sec^2\theta}\, d\theta$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \int d\theta$$

$$= \frac{2}{\sqrt{3}}\theta + C$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}(x+1)\right) + C$$