§ 7.1 Integration by Substitution

Consider the following indefinite integral

$$\int 3x^2 \cos(x^3)\, dx$$

$3x^2 \cos(x^3)$ looks like the derivative of a composite fn which comes from the chain rule.

Indeed

$$\frac{d}{dx}(\sin(x^3)) = \cos(x^3) \cdot \frac{d}{dx}(x^3)$$

$$= \cos(x^3) \cdot 3x^2$$

[Reminder: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$]

So $\sin(x^3)$ is an antid of $3x^2\cos(x^3)$ and thus

$$\int 3x^2 \cos(x^3)\, dx = \sin(x^3) + C.$$

More generally, if f, g are fns. and f has an antid. F, then by the chain rule

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x)) \cdot g'(x)$$
$$= f(g(x)) \cdot g'(x) \quad \text{as } F' = f.$$

Thus $F(g(x))$ is an antid. of $f(g(x)) \cdot g'(x)$ and so

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C.$$

Strategy is to spot the chain rule pattern in the integrand - ie. find f and g. Best bet is usually to go for the inside fn g first.

e.g in the last example, $g(x) = x^3$.

Ex

a) $\int 2e^{2x}\,dx = e^{2x} + C$ $\quad - \; g(x) = 2x$

b) $\int 2x\cos(x^2+1) = \sin(x^2+1) + C$ $\quad - \; g(x) = x^2+1$

c) $\int 2t\,e^{t^2+1}\,dt = e^{t^2+1} + C$

When the pattern isn't perfect.

Ex

$$\int x^2 \cos(x^3)\,dx.$$

Recall from before that $\frac{d}{dx}(\sin(x^3)) = 3x^2\cos(x^3)$.

We are out by a factor of 3 and we can get round this by writing

$$\int x^2\cos(x^3)\,dx = \frac{1}{3}\int 3x^2\cos(x^3)\,dx = \frac{1}{3}\sin(x^3) + C.$$

A more systematic method than the 'guess and check' method we have already seen is the <u>method of substitution</u>.

<u>Steps for the Method of Substitution</u>

1. Let w be the 'inside' <u>fn</u>

   Then $dw = w'(x)\,dx = \frac{dw}{dx} \cdot dx$.

2. Rewrite the integral as an integral in w

3. Evaluate the integral in w.

4. Convert w back to x.

Ex $\displaystyle\int 3x^2\cos(x^3)\,dx$

Let $w = x^3$, so $dw = 3x^2\,dx$.

Then

$$\int 3x^2 \overset{\cos(w)}{\cos(x^3)}\,dx = \int \cos(w)\,dw$$

dw

$$= \sin(w) + C$$

$$= \sin(x^3) + C.$$

Why substitution works

Let f, g be fns and let F be an antid. of f (as before).

We showed earlier (using the chain rule) that

$$\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C.$$

Now let $w = g(x)$, so $dw = g'(x)\,dx$.
The integral can now be written as

$$\int f(w)\,dw = F(w) + C \quad \text{as } F \text{ is an antid of } f$$

$$= F(g(x)) + C.$$

Thus

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(w)\,dw, \quad w = g(x).$$

What we gain here is that the integral on the rhs. is simpler than the one on the lhs.

Ex

$$\int t e^{t^2+1} \, dt$$

Inside fn is t^2+1.

So let $w = t^2+1$, and then

$$dw = w'(t)\,dt = 2t\,dt$$

Close to $t\,dt$ but not quite the same.

However $\frac{dw}{2} = t\,dt$

and we can rewrite the integral as

$$\int \underbrace{t\, e^{\overset{w}{(t^2+1)}} dt}_{\frac{dw}{2}} = \int e^w \cdot \frac{dw}{2} = \frac{1}{2}\int e^w \, dw$$

$$= \frac{1}{2} e^w + C$$

$$= \frac{1}{2} e^{t^2+1} + C$$

Ex. $\int x^3 \sqrt{x^4+5}\,dx$

Inside fn is x^4+5, so let

$$w = x^4 + 5$$

$$dw = 4x^3\,dx$$

Then $\frac{dw}{4} = x^3\,dx$ - matches!

Rewrite

$$\int \underbrace{x^3 \overbrace{\sqrt{x^4+5}}^{\sqrt{w}}\,dx}_{\frac{dw}{4}} = \frac{1}{4}\int \sqrt{w}\,dw$$

$$= \frac{1}{4}\int w^{\frac{1}{2}}\,dw \qquad \left[\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1\right]$$

$$= \frac{1}{4}\frac{w^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{1}{4}\cdot\frac{2}{3}w^{\frac{3}{2}} + C$$

$$= \frac{1}{6}(x^4+5)^{\frac{3}{2}} + C$$

WARNING

The only 'mismatch' in the pattern which can be fixed is being out by a multiplicative constant.

e.g. For

$$\int x^2 \sqrt{x^4+5}\, dx$$

letting $w = x^4+5$ won't work as

$$dw = 4x^3\, dx$$

and there isn't a nice way to write

$$x^2\, dx$$

in terms of w and dw only.

Ex.

$$\int e^{\cos\theta} \cdot \sin\theta \, d\theta$$

Inner fn $\quad w(\theta) = \cos\theta$

$\qquad dw = -\sin\theta \, d\theta,$

so $\quad -dw = \sin\theta \, d\theta.$

Get

$$-\int e^w \, dw = -e^w + C$$

$$= -e^{\cos\theta} + C$$

Ex.

$$\int \frac{e^t}{1+e^t} \, dt.$$

If we rewrite this slightly as

$$\int e^t \cdot \left(\frac{1}{1+e^t}\right) dt,$$

we see that the inside fn is $w(t) = 1+e^t$

Then $\quad w = 1 + e^t$

$$dw = e^t dt$$

So

$$\int \overbrace{e^t \cdot \underbrace{\frac{1}{1+e^t}}_{\frac{1}{w}} dt}^{dw} = \int \frac{1}{w} dw$$

$$= \ln|w| + C \qquad \left[\int \frac{1}{x} dx = \ln|x| + C \right]$$

$$= \ln|1 + e^t| + C$$

Ex.

$$\int \tan\theta \, d\theta = \int \frac{\sin\theta}{\cos\theta} d\theta = \int \left(\frac{1}{\cos\theta}\right) \cdot \sin\theta \, d\theta$$

Let $\quad w = \cos\theta$

$$dw = -\sin\theta \, d\theta$$

$$-dw = \sin\theta \, d\theta$$

So

$$\int \tan\theta \, d\theta = -\int \frac{1}{w} dw = -\ln|w| + C$$

$$= -\ln|\cos\theta| + C$$

$$= \ln|\sec\theta| + C, \qquad \sec\theta = \frac{1}{\cos\theta}.$$

Definite Integrals by Substitution

Ex. $\displaystyle\int_0^2 x e^{x^2}\,dx.$

Two methods

Method 1 Use substitution as before to first find an antiderivative. Then use FTOC with the original limits.

Inside fn is x^2, so let $w = x^2$

$$dw = 2x\,dx$$

$$\frac{dw}{2} = x\,dx.$$

So $\displaystyle\int x e^{x^2}\,dx = \frac{1}{2}\int e^w\,dw = \frac{e^w}{2} + C$

$$= \frac{e^{x^2}}{2} + C.$$

Thus an antid of xe^{x^2} is $\frac{e^{x^2}}{2}$ and so by FTOC

$$\int_0^2 xe^{x^2}\,dx = \left[\frac{1}{2}e^{x^2}\right]_0^2$$

$$= \frac{1}{2}e^{2^2} - \frac{1}{2}e^{0^2}$$

$$= \frac{1}{2}e^4 - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e^4 - 1)$$

<u>Method 2</u> Again use substitution, but instead of converting back to x, change the limits in x to limits in w.

Again, let $w = x^2$.

When $x = 0$ (lower limit), $w = 0$

$x = 2$ (upper limit), $w = 4$.

So

$$\int_0^2 x e^{x^2} dx = \int_0^4 \frac{e^w}{2} dw$$

$$= \left[\frac{e^w}{2} \right]_0^4$$

$$= \frac{e^4}{2} - \frac{e^0}{2}$$

$$= \frac{1}{2}(e^4 - 1) \quad \text{as before.}$$

Which method you choose is up to you.

Method 2 is usually quicker, but you need to remember to <u>CHANGE YOUR LIMITS</u>!

Ex. $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\tan^3\theta}{\cos^2\theta} \cdot d\theta$

Two possible guesses for w.

$w = \tan\theta$

$dw = \sec^2\theta \, d\theta$

$= \dfrac{1}{\cos^2\theta} \, d\theta$ ✓

$w = \sin\theta$

$dw = \cos\theta \, d\theta$

doesn't involve $\tan\theta$ X.

So stick with first choice.

Use method 2. <u>Limits.</u> when $\theta = 0$, $w = 0$

$\theta = \frac{\pi}{4}$, $w = 1$

Get

$$\int_0^{\frac{\pi}{4}} \frac{\overbrace{\tan^3\theta}^{w^3}}{\cos^2\theta} \underbrace{d\theta}_{dw} = \int_0^1 w^3 \, dw$$

$$= \left[\frac{w^4}{4} \right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}.$$

Ex.

$$\int_1^3 \frac{dx}{5-x}$$

Let $w = 5-x$, $dw = -dx$

$-dw = dx.$

Limits when $x = 1$, $w = 4$

$x = 3$, $w = 2$.

Get $\displaystyle\int_4^2 -\frac{dw}{w} = -\int_4^2 \frac{dw}{w}$

Limits wrong way round, but swapping the limits changes the sign of the integral and so allows us to get rid of the minus sign

$$= \int_2^4 \frac{dw}{w} = \Big[\ln|w|\Big]_2^4$$

$$= \ln 4 - \ln 2$$

$$= \ln\left(\frac{4}{2}\right) = \ln 2 \approx .693.$$

More Complex Substitutions

Ex. $\int \sqrt{1+\sqrt{x}}\,dx$

Again let w be the 'inside' fn, i.e.

$w = 1+\sqrt{x}$, $dw = \frac{1}{2\sqrt{x}}\,dx$, $dx = 2\sqrt{x}\,dw$

Now $w = 1+\sqrt{x}$

$w-1 = \sqrt{x}$

So $dx = 2\sqrt{x}\,dw = 2(w-1)\,dw$

Rewrite.

$$\int \sqrt{1+\sqrt{x}}\,dx = \int \sqrt{w} \cdot 2(w-1)\,dw$$

$$= 2\int (w\sqrt{w} - \sqrt{w})\,dw$$

$$= 2\int (w^{\frac{3}{2}} - w^{\frac{1}{2}})\,dw$$

$$= 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + C$$

Convert back to x.

$$= \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C$$

Ex:

$$\int (x+7)\sqrt[3]{3-2x}\ dx.$$

Let w be the inside fn.

$w = 3-2x$, $dw = -2dx$, $dx = -\frac{dw}{2}$.

Want x in terms of w

$$w - 3 = -2x$$

$$3 - w = 2x$$

$$x = \frac{3-w}{2}$$

$$= \frac{3}{2} - \frac{w}{2}$$

$$x + 7 = \frac{3}{2} - \frac{w}{2} + 7$$

$$= \frac{17}{2} - \frac{w}{2}.$$

Rewrite:

$$\int (x+7)\sqrt[3]{3-2x}\ dx = \int \left(\frac{17}{2} - \frac{w}{2}\right) \cdot \sqrt[3]{w} \cdot -\frac{dw}{2}$$

$$= -\int \left(\frac{17}{4} - \frac{w}{4}\right) w^{\frac{1}{3}}\, dw$$

$$= -\int \left(\frac{17}{4} w^{\frac{1}{3}} - \frac{w^{4/3}}{4}\right) dw$$

$$= \int \left(\frac{w^{4/3}}{4} - \frac{17}{4} w^{\frac{1}{3}}\right) dw$$

$$= \frac{1}{4} \cdot \frac{w^{7/3}}{7/3} - \frac{17}{4} \cdot \frac{1}{4/3} w^{4/3} + C$$

$$= \frac{3}{28} w^{7/3} - \frac{51}{16} w^{4/3} + C$$

Convert back to x!

$$= \frac{3}{28}(3-2x)^{7/3} - \frac{51}{16}(3-2x)^{4/3} + C$$