§ 10.5 Fourier Series

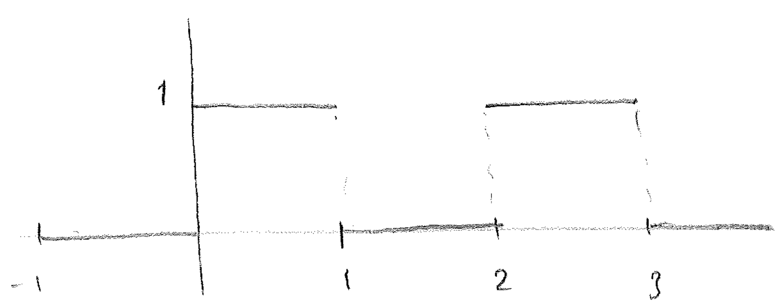
With Taylor series we approximated a fn using powers of $(x-a)$, $(x-a)^2$.

For Fourier series, we approximate a fn using sines and cosines of the type $\sin kx$, $\cos kx$.

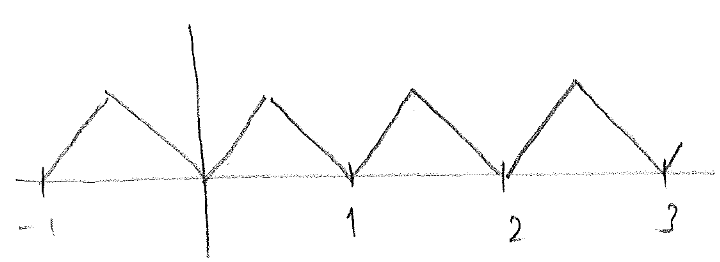
Since these fns are all periodic, we can only hope to approximate periodic fns.

e.g.

Square Wave



Triangular Wave



We will attempt to approximate f with a sum of trig. fns of the form

$$f(x) \approx F_n(x)$$

$$= a_0 + a_1\cos x + a_2\cos(2x) + a_3\cos(3x) + \cdots + a_n\cos(nx)$$

$$+ b_1\sin x + b_2\sin(2x) + b_3\sin(3x) + \cdots + b_n\sin(nx).$$

$F_n(x)$ is called the <u>Fourier polynomial of degree n</u> (even though it isn't actually a polynomial) and the numbers a_k, b_k are called the <u>Fourier coefficients</u>.

Note that since all the fns $\cos(kx)$, $\sin(kx)$ repeat every 2π, so must the original fn $f(x)$.

For such a fn there is a formula for the Fourier coefficients which gives a good approximation to f.

The Fourier Coefficients for a Periodic Function f of Period 2π

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

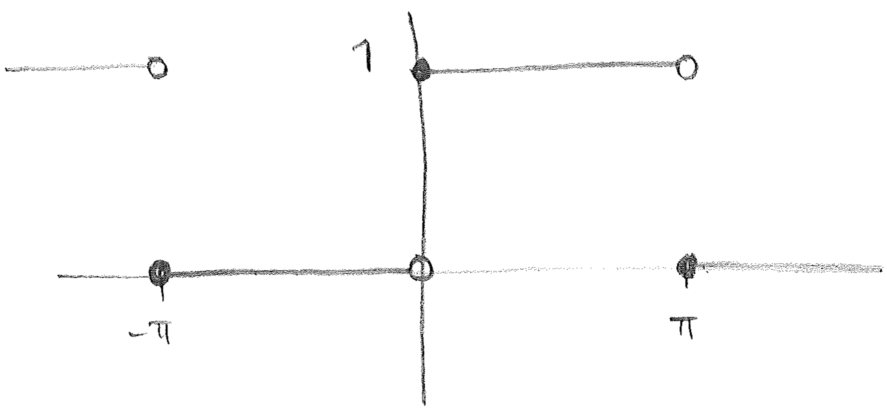
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx, \qquad k > 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\,dx, \qquad k > 0$$

Note that a_0 is simply the average value of f on $[-\pi, \pi]$.

Ex. Construct successive Fourier polynomials for the square wave fn f of period 2π given by

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$



Since a_0 is the average value on $[-\pi, \pi]$, it's reasonable to guess that $a_0 = \frac{1}{2}$. Checking gives

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi} \int_{0}^{\pi} 1\,dx = 0 + \frac{1}{2\pi}(\pi) = \frac{1}{2}.$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{2\pi} \int_{0}^{\pi} \cos(kx)\,dx$$

$$= 0 + \frac{1}{\pi} \left[\frac{\sin(kx)}{k} \right]_0^{\pi}$$

$$= 0 + \frac{1}{\pi} \left(\frac{\sin(k\pi)}{k} - \frac{\sin 0}{k} \right)$$

$$= 0.$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} 0 \, dx + \frac{1}{\pi} \int_{0}^{\pi} \sin(kx)\,dx.$$

$$= 0 + \frac{1}{\pi} \left[-\frac{1}{k} \cos(kx) \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left(-\frac{1}{k} \cos(k\pi) - \left(-\frac{1}{k} \cos 0 \right) \right)$$

$$= \frac{1}{\pi k} \left(1 - \cos(k\pi)\right)$$

Now $\cos(k\pi) = \begin{cases} -1, & k \text{ odd} \\ 1, & k \text{ even} \end{cases}$

and so

$$b_k = \begin{cases} \frac{2}{\pi k}, & k \text{ odd} \\ 0, & k \text{ even.} \end{cases}$$

From this we see that

$$F_1(x) = \frac{1}{2} + \frac{2}{\pi} \sin x$$

$$F_2(x) = \frac{1}{2} + \frac{2}{\pi} \sin x \qquad (+\, 0 \sin(2x))$$

$$F_3(x) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin(3x)$$

$$F_4(x) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin(3x) \qquad (+\, 0 \sin(4x))$$

$$F_5(x) = \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin(3x) + \frac{2}{5\pi} \sin(5x).$$

etc.

Using Maple, we can see how the Fourier polynomials give better and better approximations to the square wave.

Note that as $n \to \infty$, the approximation given by F_n gets better everywhere (except for the points of discontinuity).

Note also that this is different to the situation with Taylor polynomials where the approximation is usually just on an interval (i.e. locally).

As $n \to \infty$, the Fourier polynomials F_n give us the <u>Fourier Series</u> for f.

On the interval $[-\pi, \pi]$, the Fourier series for f is given by

$$a_0 + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + \cdots$$
$$+ b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + \cdots$$

where a_k and b_k are the Fourier coefficients whose formulae were given earlier.

If f is differentiable, then this Fourier series converges to $f(x)$ and we have

$$f(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + \cdots$$
$$+ b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + \cdots$$

The term

$$a_1 \cos x + b_1 \sin x$$

of period 2π is called the <u>fundamental harmonic</u> of f while the term

$$a_k \cos(kx) + b_k \sin(kx)$$

of period $\frac{2\pi}{k}$ is called the <u>kth harmonic</u> of f.

e.g In the last example, the fundamental harmonic was

$$\frac{2}{\pi} \sin x$$

while the k-th harmonic was

$$\begin{cases} \frac{2}{k\pi} \sin(kx), & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

Energy and the Energy Theorem

We define the energy E of a periodic fn f of period 2π by

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx.$$

One can check that for the kth harmonic

$$a_k \cos(kx) + b_k \sin(kx)$$

the energy is given by

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (a_k \cos(kx) + b_k \sin(kx))^2\,dx = a_k^2 + b_k^2$$

$$= A_k^2$$

Where $A_k = \sqrt{a_k^2 + b_k^2}$ is called the amplitude of the k-th harmonic.

The energy of the constant term a_0 in the Fourier series is given by

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} a_0^2 \, dx = 2a_0^2$$

and so we define the amplitude A_0 of the constant term by

$$A_0 = \sqrt{2A_0^2} = \sqrt{2} \cdot a_0 .$$

The big result is then that for a fn f which is periodic of period 2π

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 dx = A_0^2 + A_1^2 + A_2^2 + A_3^2 + \cdots$$

$$= 2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \cdots$$
$$+ b_1^2 + b_2^2 + b_3^2 + \cdots$$

This is the so-called energy theorem (also known as Parseval's identity).

It can be thought of as an infinite-dimensional version of Pythagoras' theorem.

e.g. In the example with the square wave

$$E = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{\pi}\int_{0}^{\pi} 1^2\,dx = 1.$$

$$a_0 = \frac{1}{2}, \quad a_k = 0, \quad k \geq 1$$

$$b_k = \begin{cases} \frac{2\pi}{k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

Thus $A_0^2 = 2a_0^2 = 2\left(\frac{1}{2}\right)^2 = \frac{1}{2}.$

$$A_k^2 = \begin{cases} \dfrac{4}{\pi^2 k^2}, & k \text{ odd} \\ 0, & k \text{ even.} \end{cases}$$

Then

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 + \cdots$$

$$= \frac{1}{2} + \frac{4}{\pi^2 1^2} + 0 + \frac{4}{\pi^2 \cdot 3^2} + \cdots$$

$$= \frac{1}{2} + \frac{4}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right)$$

$$= \frac{1}{2} + \frac{4}{\pi^2}\left(\frac{\pi^2}{8}\right)$$

$= \frac{1}{2} + \frac{1}{2} = 1$ using the fact that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Functions which Have Periods Other than 2π

Suppose $f(x)$ has period b.

If we let

$$x = \frac{bt}{2\pi},$$

then as x ranges over the interval $[-b/2, b/2]$, t will range over the interval $[-\pi, \pi]$.

Now set

$$g(t) = f\left(\frac{bt}{2\pi}\right) = f(x).$$

g is then periodic of period 2π and we can find the Fourier series for g in the usual way.

$$g(t) = a_0 + a_1 \cos t + a_2 \cos(2t) + a_3 \cos(3t) + \cdots$$
$$+ b_1 \sin t + b_2 \sin(2t) + b_3 \sin(3t) + \cdots$$

Thus, since $t = \frac{2\pi x}{b}$

$$f(x) = g(t) = a_0 + a_1 \cos\left(\frac{2\pi x}{b}\right) + a_2 \cos\left(\frac{4\pi x}{b}\right) + a_3 \cos\left(\frac{6\pi x}{b}\right) + \cdots$$
$$+ b_1 \sin\left(\frac{2\pi x}{b}\right) + b_2 \sin\left(\frac{4\pi x}{b}\right) + b_3 \sin\left(\frac{6\pi x}{b}\right) + \cdots$$

What about the Fourier coefficients?

Well.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt = \frac{1}{2\pi} \int_{-b/2}^{b/2} f(x) \cdot \frac{2\pi}{b}\, dx$$

$$= \frac{1}{b} \int_{-b/2}^{b/2} f(x)\, dx$$

as $g(t) = f(x)$ and $t = \frac{2\pi x}{b}$

so $dt = \frac{2\pi}{b}\, dx$.

For $k \geq 1$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(kt)\, dt$$

$$= \frac{1}{\pi} \int_{-b/2}^{b/2} f(x) \cdot \cos\left(\frac{2\pi k x}{b}\right) \cdot \frac{2\pi\, dx}{b}$$

again, $dt = \frac{2\pi dx}{b}$

$$= \frac{2}{b} \int_{-b/2}^{b/2} f(x) \cos\left(\frac{2\pi k x}{b}\right) dx.$$

Similarly, for $k \geq 1$.

$$b_k = \frac{2}{b} \int_{-b/2}^{b/2} f(x) \sin\left(\frac{2\pi k x}{b}\right) dx.$$

To summarize.

If f is periodic of period b, then the Fourier series for f on $[-\frac{b}{2}, \frac{b}{2}]$ is given by

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{2\pi k x}{b}\right) + b_k \sin\left(\frac{2\pi k x}{b}\right)\right)$$

where

$$a_0 = \frac{1}{b} \int_{-\frac{b}{2}}^{\frac{b}{2}} f(x)\,dx,$$

and for $k \geqslant 1$,

$$a_k = \frac{2}{b} \int_{-b/2}^{b/2} f(x) \cos\left(\frac{2\pi k x}{b}\right) dx,$$

$$b_k = \frac{2}{b} \int_{-b/2}^{b/2} f(x) \sin\left(\frac{2\pi k x}{b}\right) dx.$$

Ex. $f(x) = x^2$ on $[-1, 1]$

Here $b = 2$, $b/2 = 1$ and

$$a_0 = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{3} - \left(\frac{-1}{3}\right)\right) = \frac{1}{3}.$$

For $k \geq 1$

$$a_k = \frac{2}{2}\int_{-1}^{1} x^2 \cos\left(\frac{2\pi k x}{2}\right) dx$$

$$= \int_{-1}^{1} x^2 \cos(\pi k x)\,dx$$

by parts

$u = x^2$, $dv = \cos(\pi k x)\,dx$

$du = 2x\,dx$, $v = \frac{\sin(\pi k x)}{\pi k}$

$$= \left[\frac{x^2 \sin(\pi k x)}{\pi k}\right]_{-1}^{1} - \frac{2}{\pi k}\int_{-1}^{1} x \sin(\pi k x)\,dx$$

by parts again

$u = x$, $dv = \sin(\pi k x)\,dx$

$du = dx$, $v = -\frac{\cos(\pi k x)}{\pi k}$

$$= \left[\frac{x^2 \sin(\pi k x)}{\pi k}\right]_{-1}^{1} - \frac{2}{\pi k}\left\{\left[\frac{-x\cos(\pi k x)}{\pi k}\right]_{-1}^{1} - \int_{-1}^{1} -\frac{\cos(\pi k x)}{\pi k}\,dx\right\}$$

$$= \left[\frac{x^2 \sin(\pi k x)}{\pi k} \right]_{-1}^{1} - \frac{2}{\pi k} \left\{ \left[\frac{-x\cos(\pi k x)}{\pi k} \right]_{-1}^{1} + \int_{-1}^{1} \frac{\cos(\pi k x)}{\pi k}\, dx \right\}$$

$$= \left[\frac{x^2 \sin(\pi k x)}{\pi k} \right]_{-1}^{1} - \frac{2}{\pi k} \left\{ \left[\frac{-x\cos(\pi k x)}{\pi k} \right]_{-1}^{1} + \left[\frac{\sin(\pi k x)}{\pi^2 k^2} \right]_{-1}^{1} \right\}$$

$$= \frac{1^2 \sin(\pi k)}{\pi k} - \frac{(-1)^2 \sin(-\pi k)}{\pi k}$$

$$- \frac{2}{\pi k} \left\{ \frac{-1\cos(\pi k)}{\pi k} - \left(\frac{-(1)\cos(-\pi k)}{\pi k} \right) + \frac{\sin(\pi k)}{\pi^2 k^2} - \frac{\sin(-\pi k)}{\pi^2 k^2} \right\}$$

Now $\sin(\pi k) = \sin(-\pi k) = 0$

while $\cos(\pi k) = \cos(-\pi k) = (-1)^k$

and so we get

$$a_k = 0 - 0$$

$$- \frac{2}{\pi k} \left\{ -\frac{2(-1)^k}{\pi k} + 0 - 0 \right\}$$

$$= \frac{4}{\pi^2 k^2} (-1)^k.$$

Now

$$b_k = \frac{2}{2}\int_{-1}^{1} x^2 \sin(\pi k x)\,dx.$$

We could integrate this by parts as before, but it is much simpler to observe that as the integrand is an <u>odd</u> fn and we are integrating over an interval which is symmetric about 0, we must have

$$b_k = 0.$$

Thus we have

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}\cos(\pi k x), \qquad -1 \le x \le 1.$$

Note that in this example which was an even fn, there were no sine terms in the Fourier series while for the last example (the square wave), there were no cosine terms.

This is an example of the following general feature

- The Fourier series on $[-b/2, b/2]$ for a fn which is even contains no sine terms.

- The Fourier series on $[-b/2, b/2]$ for a fn which is either odd or a vertically shifted odd fn contains no cosine terms.

Finally, some entertainment!

If we let $x = 1$ in

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(\pi k x)$$

we get

$$1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \cos(\pi k)$$

and since $\cos(\pi k) = (-1)^k$

$$1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} (-1)^k$$

$$1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{2k}}{k^2}$$

So

$$1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

$$\frac{2}{3} = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2}{3} \cdot \frac{\pi^2}{4} = \frac{\pi^2}{6}$$

So using Fourier series we have proved that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad !$$