§ 8.5 Applications to Physics

Many problems in physics are solved by calculating integrals.

Work/Energy

If a constant force F is applied to move an object a distance d, we define the work W done to be the product

$$W = F \cdot d.$$

Units

In metric (SI) units, the unit of force is the Newton, N $1\,N = 1\,kg\,m/s^2$.

The unit of work/energy is the Joule J

$$1\,J = 1\,Nm = 1\,kg\,m^2/s^2.$$

In imperial (English) units, the unit of force is the pound, lb and the unit of work is the foot pound ft lb.

Ex. If it takes 50 lbs of force to move a box 10 ft across the floor of my basement, the work I do is

$$50 \times 10 = 500 \text{ ft lbs.}$$

Weight

Weight is the force exerted on an object due to the Earth's gravity acting on its mass.

In metric units, the weight of an object of mass m kg is

$$mg \quad \text{Newtons}$$

where $g = 9.81 \, m/s^2$ is the acceleration due to gravity at the Earth's surface.

In English units, the pound is a unit of <u>force</u> (not of mass), so when we say an object is 10 lbs, we mean that the weight is 10 lbs, ie. the Earth's gravitational field exerts 10 lbs of force on it (n.b. one doesn't often talk of mass in English units - can you think why this might be so?).

A good example of the difference between mass and weight is to imagine standing on the surface of the moon whose gravity is six times weaker than Earth's. You <u>weight</u> is $\frac{1}{6}$ that of your weight on Earth, although your <u>mass</u> is unchanged.

<u>Ex</u>. I lift a 2 kg book from the floor onto a shelf 2 m above the floor. The amount of work I do is

$$2\,kg \times 9.8\,m/s^2 \times 2\,m = 38.4\,J.$$

Work against a variable force.

If the force varies with the distance x, the expression

$$W = F \cdot d$$

is replaced with the integral

$$W = \int_a^b F(x)\,dx .$$

Measures the work done using a variable force $F(x)$ to move the object (in a straight line) from $x = a$ to $x = b$.

__Ex__ Hooke's Law.

If a spring has natural length L and we extend it to a lenghth L', then the force exerted by the spring (i.e. the tension) is given by

$$F = k(L' - L)$$

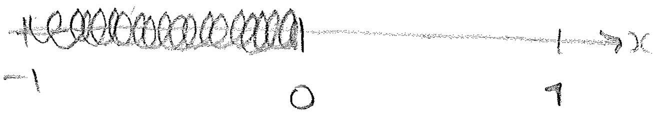
where k is a constant of proportionality called the __spring constant__.

The bigger k is, the stiffer the spring.

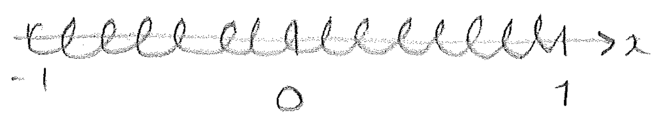
e.g. Suppose we have a spring of natural length 1m and spring constant 4 which we then extend to 2m.

Place the spring on the x-axis so that in the unextended position, the right end of the spring sits at $x=0$.

Unextended



Extended



The force exerted when the right end of the spring sits at x $(0 \leq x)$ is then

$$k(l' - l)$$

$$= k(1 + x - 1)$$

$$= kx$$

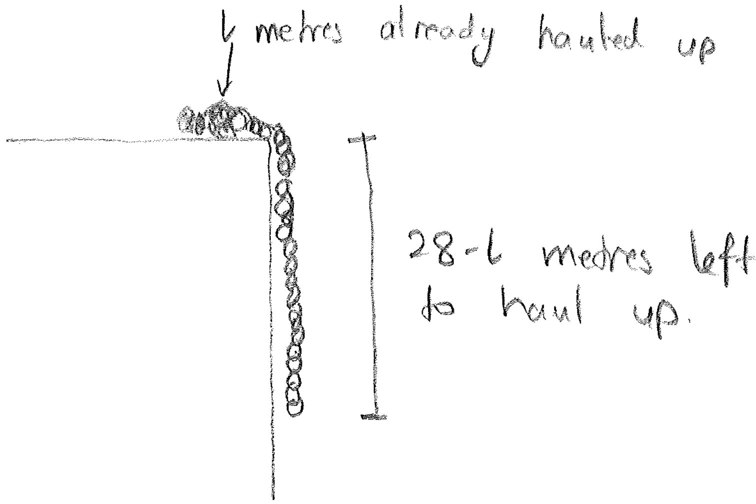
$$= 4x \qquad \text{as} \qquad k = 4.$$

The work done is then

$$W = \int_a^b F(x)\,dx = \int_0^1 4x\,dx = \left[2x^2\right]_0^1 = 2\,\text{J}.$$

Ex A 28m uniform chain with a linear density λ of $2\,kg/m$ is dangling from the roof of a building. How much work is needed to pull the chain up onto the top of the building.

Let l represent the amount of chain already hauled up.



At this pt $28-l$ metres are left to haul up and this amount of chain has mass

$$2\,kg/m \times (28-l)\,m = 56-2l \text{ kg}$$

and weight

$$(56-2l)\cdot 9.8 \text{ Newtons.}$$

The work done to haul this piece of chain up a short distance Δl is then approx the weight $\times \Delta l$, ie.

$$\Delta W \approx (56 - 2l) \cdot 9.8 \, \Delta l \quad \text{J.}$$

Now add all these contributions ΔW to get

$$W \approx \sum (56 - 2l)(9.8) \, \Delta l$$

As $\Delta l \to 0$, this Riemann sum becomes an integral and we have

$$W = \int_0^{28} (56 - 2l)(9.8) \, dl$$

$$= 9.8 \left[56l - l^2 \right]_0^{28}$$

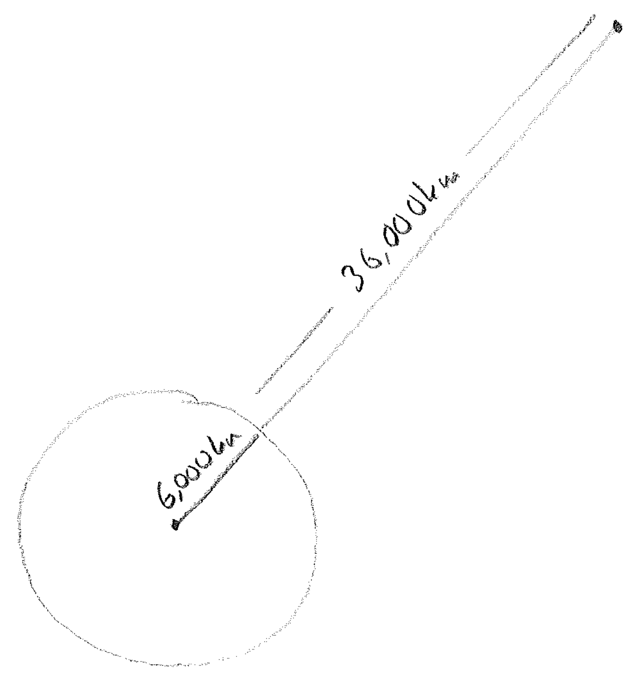
$$= 7{,}683.2 \quad \text{J.}$$

N.b. The book solves this differently by first splitting up the chain and then calculating the work needed to raise each small piece to the top.

Ex The Space Elevator

A space elevater connects a pt. on the equater with geostatinary orbit which is approx 36,000 km above that pt.

Calculate the amount of work the elevater has to do in order to lift a 70kg person from the equater to geostatinary orbit. If the elevater is powered by electricity which costs 10¢ / kWhr, how much does this cost?



The gravitational attraction (i.e. weight) of the person is given by

$$\frac{G M_E \cdot 70}{r^2} \approx \frac{2.79 \times 10^{16}}{r^2} \text{ N}$$

where $G = 6.674 \times 10^{-11}$ Nm^2/kg^2 is the gravitational constant, $M_E = 5.97 \times 10^{24}$ kg is the mass of the Earth and r is the distance from the person to the centre of the Earth.

The work required to move the person a small distance Δr is then approx.

$$\Delta W \approx \text{force} \times \text{distance} = \frac{2.79 \times 10^{16}}{r^2} \Delta r$$

and the total work is then approx

$$\sum \Delta W = \sum \frac{2.79 \times 10^{16}}{r^2} \Delta r$$

As the person travels on the elevater, r goes from about 6000 km to about 36000 + 6000 = 42000 km, i.e. from 6×10^6 m to 4.2×10^7 m.

Thus as $\Delta r \to 0$, we get

$$W = \int_{6\times10^6}^{4.2\times10^7} \frac{2.79\times10^{16}}{r^2}\, dr$$

$$= 2.79\times10^{16} \int_{6\times10^6}^{4.2\times10^7} \frac{1}{r^2}\, dr$$

$$= 2.79\times10^{16} \left[-\frac{1}{r}\right]_{6\times10^6}^{4.2\times10^7}$$

$$\approx 3.98 \times 10^9 \text{ J.}$$

Now the unit of power in the metric system is the Watt W where

$$1\,W = 1\,J/s.$$

$$1\,kW\,hr = 1000\,W \text{ for } 3600\,s$$

$$= 3.6 \times 10^6\,J.$$

Thus the amount of energy required in kWhr is

$$\frac{3.98 \times 10^9}{3.6 \times 10^6}$$

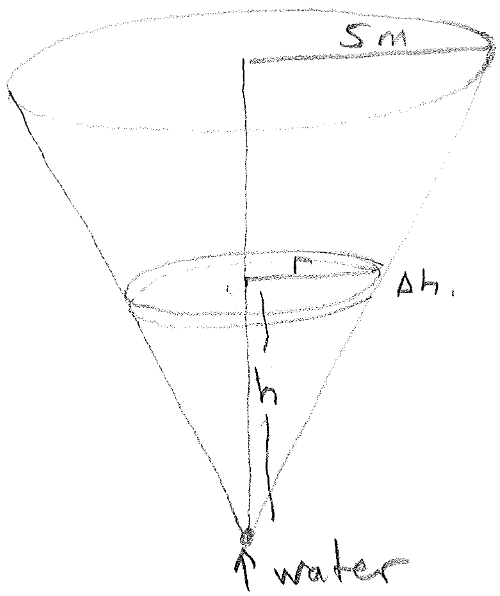
$$\approx 1107\ kWhr.$$

The cost is then $1107 \times .10 \approx \$110.66$!

Note that we have neglected factors such as the rotation of the Earth, air resistance, friction and the efficiency of the elevator, not to mention the cost of building it!

Q. So why does it cost about \$10,000/lb to send things into LEO today?

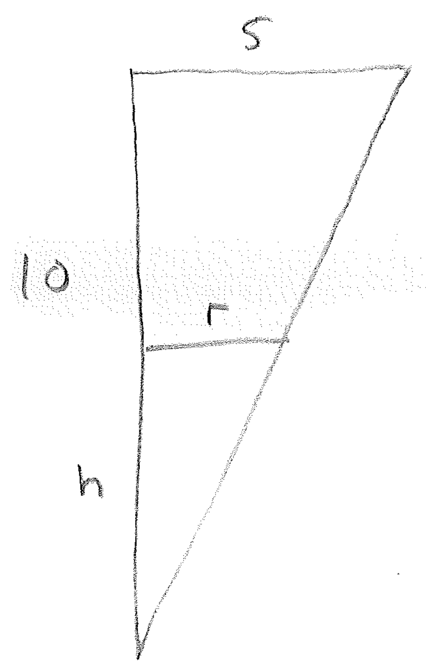
Ex. A conical tank as shown is filled with water from its base. If the density of water is $1000\,kg/m^3$, find the amount of work required to completely fill the tank.



Let h be the depth of water in the tank. The top of the water is approx. a cylinder of radius r and thickness Δh. and the volume of this piece of water is then approx

$$\pi r^2 \Delta h.$$

By similar triangles



$$\frac{r}{h} = \frac{5}{10} = \frac{1}{2}$$

$$r = h/2$$

Thus the volume of this slice of water is approx

$$\pi\left(\frac{h}{2}\right)^2 \Delta h = \frac{\pi h^2}{4}\Delta h.$$

The mass is then approx.

$$1000\,\frac{\pi h^2}{4}\Delta h = 250\pi h^2 \Delta h$$

while the weight is

$$(9.81)(250\pi h^2)\Delta h$$

To fill the tank, this slice of water needs to be raised from the base of the tank to a height h and so the work done is approx

$$\Delta W = (9.81)(250\pi h^2)\Delta h \cdot h$$

$$= (9.81)(250\pi h^3)\Delta h.$$

Summing up we get

$$W \approx \sum 9.81(250\pi h^3)\Delta h$$

and in the limit as $\Delta h \to 0$

$$W = \int_0^{10} (9.81)(250\pi h^3)\, dh$$

$$= (9.81)(250\pi)\int_0^{10} h^3\, dh$$

$$= (9.81)(250\pi)\left[\frac{h^4}{4}\right]_0^{10}$$

$$= (9.81)(250\pi)\left(\frac{10{,}000}{4}\right)$$

$$\approx 1.926 \times 10^{7} \text{ J}$$

or 19.26 MJ.

⌜A person using a steady 250W of power to operate a hand pump would take about 21.4 hours to fill this tank.⌟

Force and Pressure

Recall that pressure is force per unit area. If a constant pressure P acts on a surface of area A, then the total force on the surface is

$$F = P \cdot A$$

If the pressure varies over the surface, we need integration to calculate the total force.

In metric units, the unit of pressure is the Pascal Pa, where

$$1 \text{ Pa} = 1 \text{ N}/\text{m}^2.$$

In imperial (English) units, pressure is usually measured in lbs/ft^2 or lbs/in^2 (psi).

A common example of variable pressure is hydrostatic pressure from being immersed in a liquid such as water.

Hydrostatic pressure acts equally in all directions and at a depth h below the surface the pressure is given by

$$P = \rho g h$$

where ρ is the density of the fluid and g is the acceleration due to gravity.

For water

$$P = 1000(9.8)h \text{ Pa}$$

in metric units as the density of water is $1000 \text{kg}/m^3$. In imperial units, $\rho = 62.4 \text{ lbs}/ft^3$ and

$$P = (62.4)h \text{ lbs}/ft^2$$

Note that in this case, we don't need to multiply by g ($= 32 ft/s^2$) as this has already been done.

Ex. Calculate the hydrostatic force on one side of a plate of the Titanic at the bottom of the ocean if the ocean is 12,500 ft deep and the plate is a square 100 ft on a side when

a) the plate is lying horizontally

b) the plate is standing vertically.

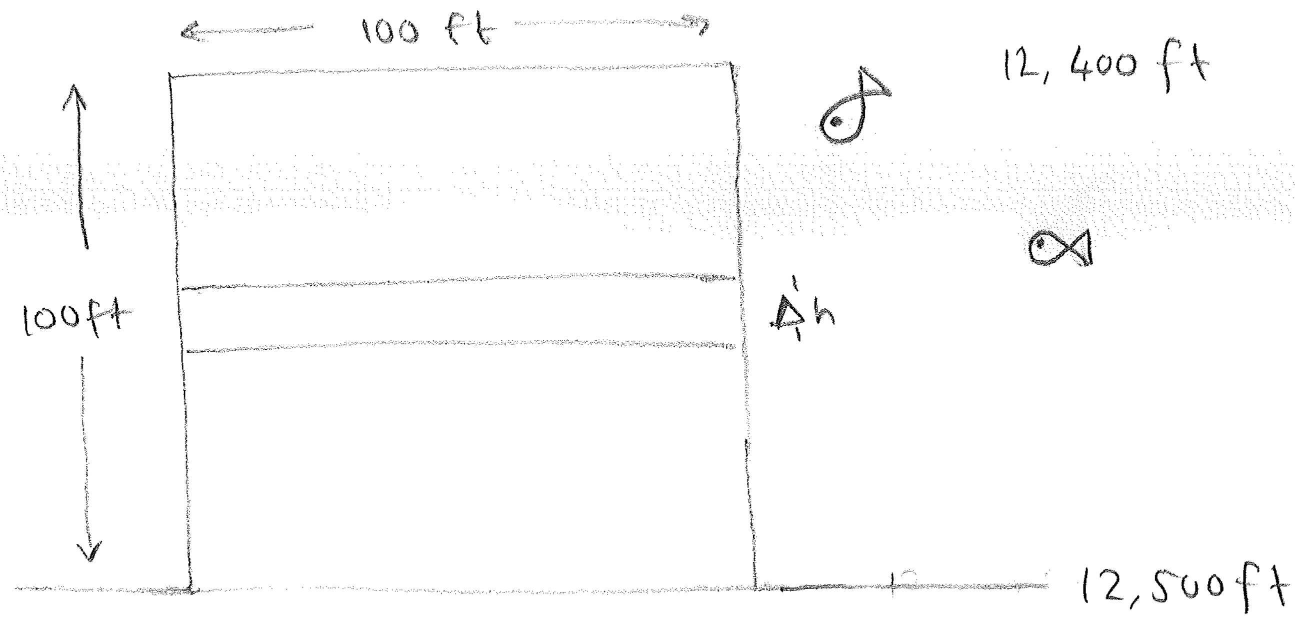
a) Since the pressure exerted depends only on the depth, in this case it is constant and the total force is

$$F = \text{Pressure} \times \text{Area}$$

$$= (62.4)(12{,}500) \times (100)^2$$

$$= 7.8 \times 10^9 \text{ lbs.}$$

b) In this case, we slice the plate horizontally



On a thin horizontal strip of area $100\,\Delta h$, the pressure is approx constant with value

$$(62.4)h$$

and so the force on the strip is approx.

$$\Delta F = ((62.4)h)\cdot(100\,\Delta h)$$

Summing up gives an approx. of the total force

$$F \approx \sum (62.4)h \cdot 100\,\Delta h$$

As we let $\Delta h \rightarrow 0$, this Riemann sum becomes an integral and

$$F = \int_{12,400}^{12,500} (62.4)h \cdot 100\, dh$$

$$= 6240 \int_{12,400}^{12,500} h\, dh$$

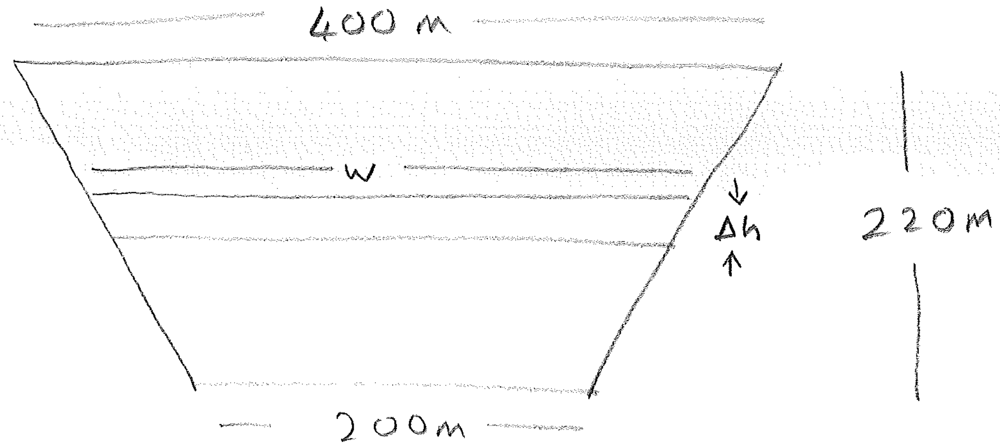
$$= 6240 \left[\frac{h^2}{2} \right]_{12,400}^{12,500}$$

$$= \frac{6240}{2} \left((12,500)^2 - (12,400)^2 \right)$$

$$= 7.7 \times 10^9 \text{ lbs}$$

Explain why this answer is slightly smaller than that in a)!

Ex A dam similar to the Hoover dam is in the shape of a trapezoid as below.



If the dam is full to the brim calculate the total force exerted by the water on the dam.

Divide the dam into horizontal strips of width w and height Δh.

The width w depends on the height h and is clearly a linear fn of h.

ie $\quad w(h) = ah + b$

Also when $h = 0$, $w = 400$ and
when $h = 220$, $w = 200$

ie $$a(0) + b = 400 \quad (1)$$

$$a(220) + b = 200 \quad (2)$$

(1) $\Rightarrow$ $b = 400$

Subst into (2) gives

$$220a + 400 = 200$$

$$220a = -200$$

$$a = -\frac{200}{220} = -\frac{10}{11}.$$

So $w = 400 - \frac{10}{11}h.$

The area of the small strip is then approx

$$\left(400 - \frac{10}{11}h\right)\Delta h$$

while the water pressure on this strip is approx constant with value

$$(1000)(9.8)h.$$

The force on this strip is then approx

pressure $\times$ area

$$\Delta F \approx (1000)(9.8)h \cdot (400 - \tfrac{10}{11}h)\Delta h$$

The total force, F, is then approx

$$F \approx \sum \Delta F = \sum (1000)(9.8)(400 - \tfrac{10}{11}h)\Delta h$$

and as $\Delta h \to 0$ we get that

$$F = \int_0^{220} (1000)(9.8)h(400 - \tfrac{10}{11}h)\,dh$$

$$= 9{,}800 \int_0^{220} (400h - \tfrac{10}{11}h^2)\,dh$$

$$= 9{,}800 \left[200h^2 - \tfrac{10}{33}h^3\right]_0^{220}$$

$$\approx 6.32 \times 10^{10}\ \text{N.}$$