§ 7.5, 7.6 Approximating Definite Integrals

Midpoint Rule, Trapezoidal Rule, Simpson's Rule

We consider an integrable fn f defined on a closed interval $[a,b]$.

As before we divide $[a,b]$ into n equal subintervals of with $\Delta x = \frac{b-a}{n}$ which are pairwise disjoint except for endpts.

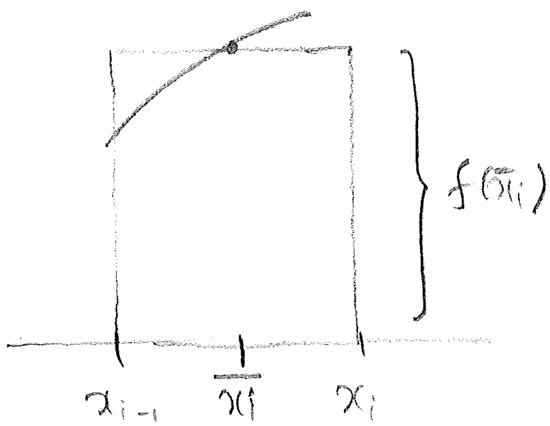
The ith interval is $[x_{i-1}, x_i]$ where $x_i = a + i\Delta x = a + i\frac{(b-a)}{n}$.

Idea of this section is to find schemes which give us a better approximation to the definite integral than the left- and right-hand sums do.

Midpoint Rule

Let $\overline{x_i}$ be the midpoint of $[x_{i-1}, x_i]$

ie. $\overline{x_i} = \dfrac{x_{i-1} + x_i}{2}$



Picture suggests that a rectangle of height $f(\overline{x_i})$ will give a better approx. to the true area under the curve than one of height $f(x_{i-1})$ (left-hand sum) or height $f(x_i)$ (right-hand sum).

Area of rectangle is $f(\overline{x_i})\,\Delta x$

Add the contributions from all the rectangles together to get the Riemann sum

$$\sum_{i=1}^{n} f(\overline{x_i})\,\Delta x.$$

For general f on $[a,b]$, let

LEFT(n), RIGHT(n) and MID(n)

denote the left- right- and midpt. approximations with n subintervals.

Ex For $\int_1^2 \frac{1}{x}\,dx$, find LEFT(2), RIGHT(2) and MID(2).

$n = 2 \Rightarrow 2$ subintervals, $[1, 1.5]$ and $[1.5, 2]$.

$$\Delta x = \frac{2-1}{2} = .5$$

$$\text{LEFT}(2) = f(1)(.5) + f(1.5)(.5)$$

$$= \frac{1}{1}(.5) + \frac{1}{1.5}(.5) \approx 0.8333$$

$$\text{RIGHT}(2) = f(1.5)(.5) + f(2)(.5)$$

$$= \frac{1}{1.5}(.5) + \frac{1}{2}(.5) \approx 0.5833$$

$$\text{MID}(2) = f(1.25)(.5) + f(1.75)(.5)$$

$$= \frac{1}{1.25}(.5) + \frac{1}{1.75}(.5) \approx 0.6857$$

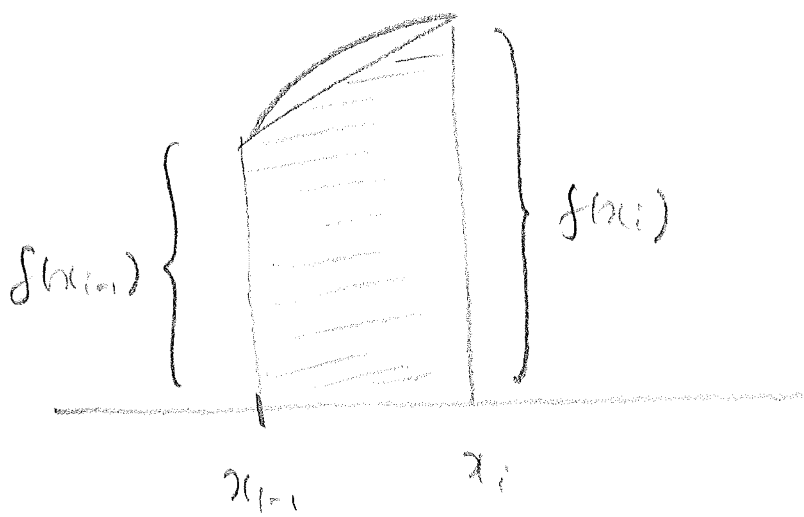
Note that the true value of this integral is

$$\int_1^2 \frac{1}{x}\,dx = \Big[\ln x\Big]_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.6931.$$

NOTE MID(2) is much closer to the true value of this integral than LEFT(2) or RIGHT(2).

The Trapezoid Rule

Approximate the area under the curve over $[x_{i-1}, x_i]$ by a trapezoid as shown



Area of trapezoid is the average of those for the left- and right hand sums.

is $\dfrac{f(x_{i-1}) + f(x_i)}{2} \Delta x$

If we add all the bits from each interval together and let TRAP(n) be the resulting sum, then

$$\text{TRAP}(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}$$

Ex For $\int_1^2 \frac{1}{x}\,dx$, we saw that

$\text{LEFT}(2) \approx 0.8333$

$\text{RIGHT}(2) \approx 0.5833$

Thus $\text{TRAP}(2) \approx \frac{0.8333 + 0.5833}{2} = 0.7083$

NOTE This is closer to the true value of $\ln 2$ than LEFT(2), RIGHT(2). However MID(2) is better by roughly a factor of 2.

Under- or Over-Estimate?

Something we already saw in Math 141

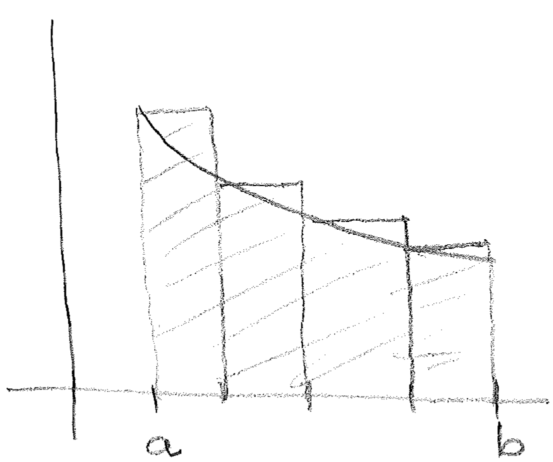
- If f is increasing on $[a,b]$ then

$$\text{LEFT}(n) \leq \int_a^b f(x)\,dx \leq \text{RIGHT}(n)$$

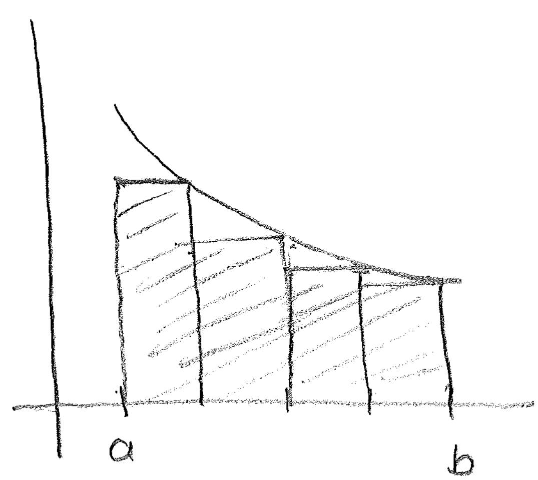
- If f is decreasing on $[a,b]$ then

$$\text{RIGHT}(n) \leq \int_a^b f(x)\,dx \leq \text{LEFT}(n).$$

PICTURE for f decr.



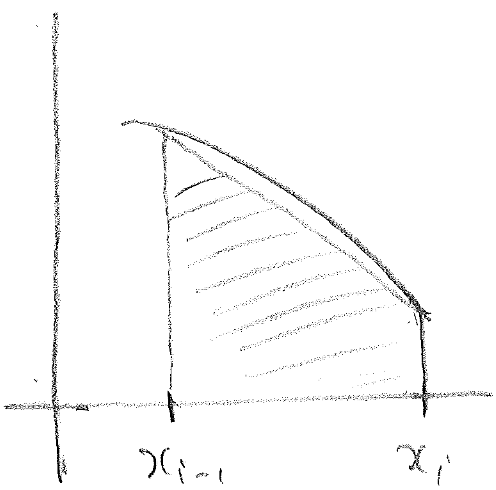
LEFT (n)



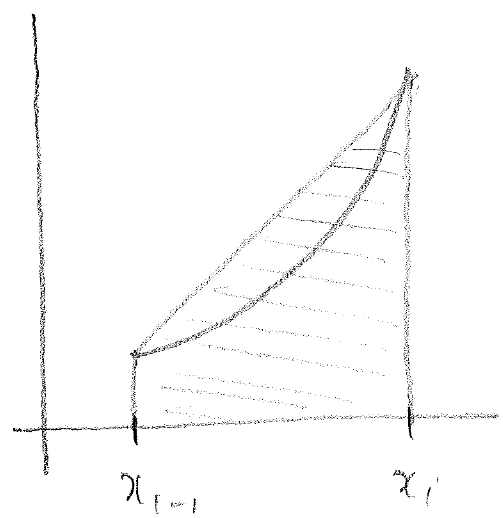
RIGHT(n).

For the midpoint and trapezoid rules, whether we get an under- or over-estimate is determined not by whether f is incr/decr but whether f is concave up or concave down.

For the trapezoid rule, the picture is as follows

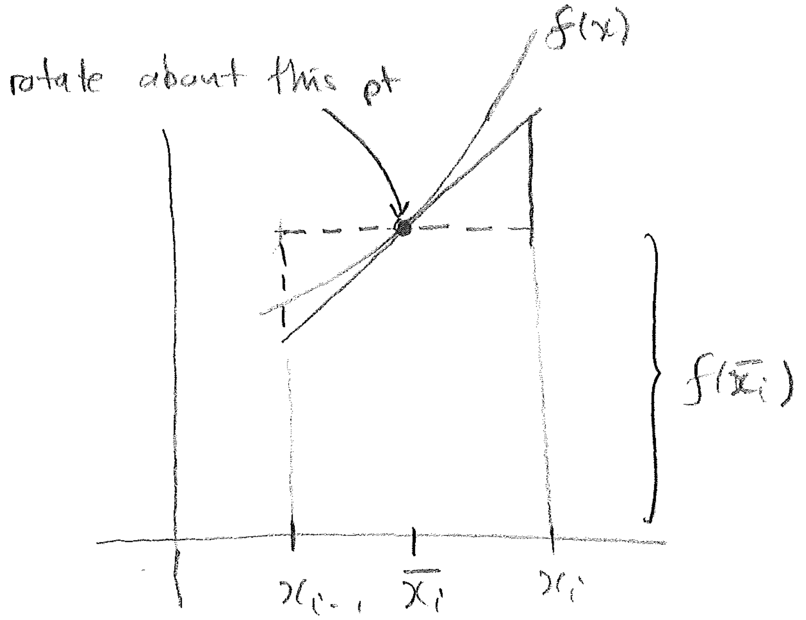


f concave down: trapezoid underestimates



f concave up: trapezoid overestimates.

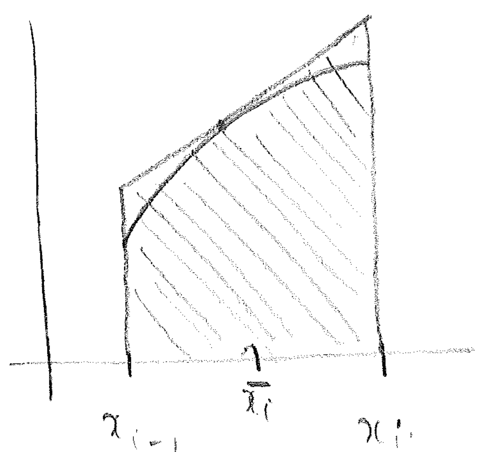
For the corresponding picture for the midpoint rule, we first need to make the little rectangles of height $f(\bar{x}_i)$ into trapezoids of the same area.



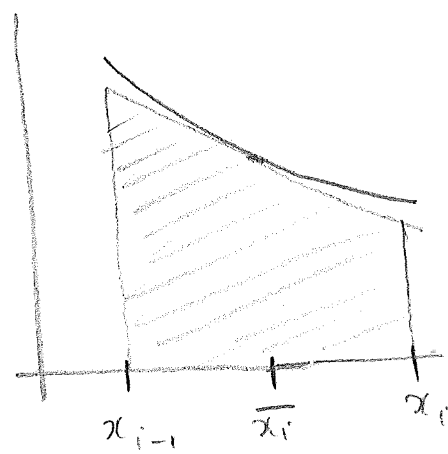
Slope of top of trapezoid is that of the tangent to the curve at $(\bar{x}_i, f(\bar{x}_i))$.

<u>**NOTE**</u>: This is <u>**NOT**</u> the same trapezoid as for the trapezoid rule.

Using these trapezoids, the picture is then



f concave down:
midpoint overestimates



f concave up:
midpoint underestimates

What we have found:

- If f is concave down on $[a,b]$, then

$$\mathrm{TRAP}(n) \leq \int_a^b f(x)\,dx \leq \mathrm{MID}(n).$$

- If f is concave up on $[a,b]$, then

$$\mathrm{MID}(n) \leq \int_a^b f(x)\,dx \leq \mathrm{TRAP}(n).$$

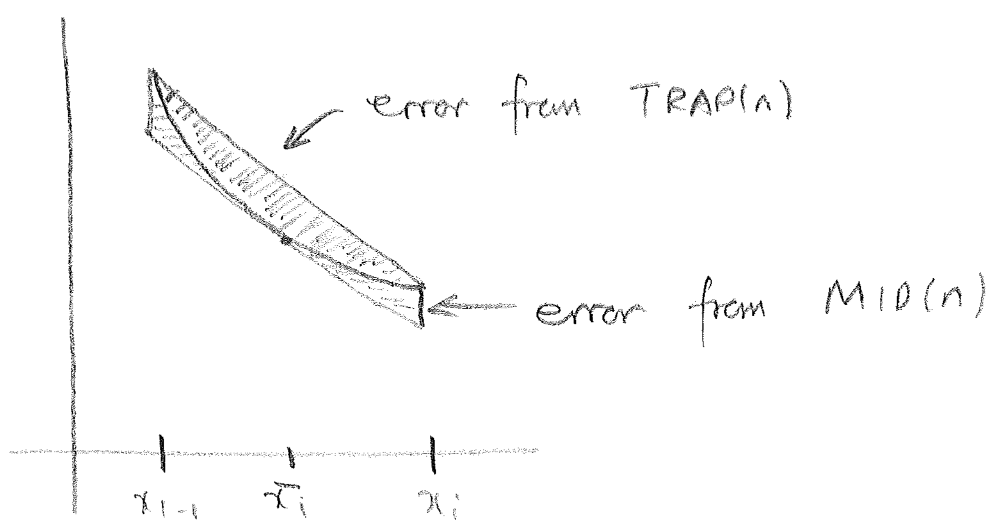
If we do some computer experiments (e.g using Maple or Mathematica), we find the following.

- The error for both MID(n) and TRAP(n) behaves like $\frac{1}{n^2}$.

  e.g. Doubling n, decreases the error by about 4.

- The error for TRAP(n) is about twice that for MID(n).

- The errors for MID(n) and TRAP(n) have opposite signs (as we'd expect).



This suggests that if we take the weighted average

$$\frac{2\,\mathrm{MID}(n) + \mathrm{TRAP}(n)}{3},$$

then the errors should nearly cancel out.

This leads to Simpson's Rule and we call the above average SIMP(n).

One can calculate that

$$\mathrm{SIMP}(n) = \frac{\Delta x}{3}\Big[f(x_0) + 4f(\bar{x}_1) + 2f(x_1) + 4f(\bar{x}_2) + 2f(x_2) + \cdots$$
$$\cdots + 2f(x_{n-1}) + 4f(\bar{x}_n) + f(x_n)\Big]$$

Some final remarks ....

- LEFT(n) and RIGHT(n) can be thought of as attempts to approximate the graph of f using horiz. & vertical lines (like a computer screen).

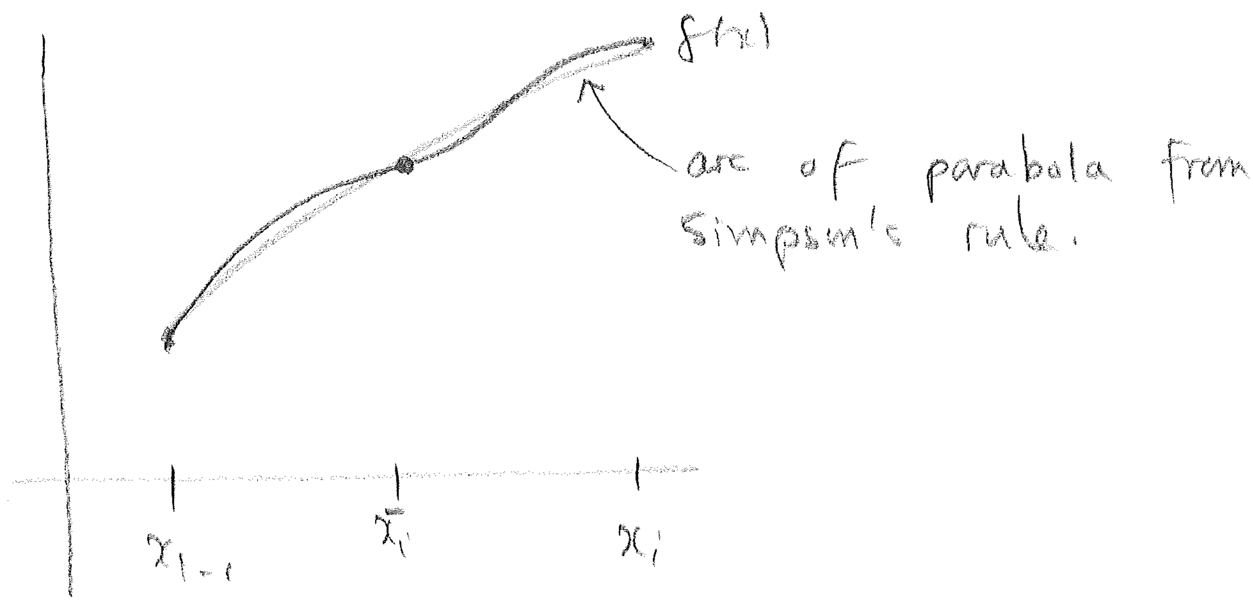
  The error behaves like $\frac{1}{n}$ and the more f decreases or increases (ie the bigger f' is), the worse the approximation will be.

- MID(n) and TRAP(n) can be thought of as attempts to approximate the graph of f using (sloping) straight lines.

  The error behaves like $\frac{1}{n^2}$ and the more the graph of f bends (ie the bigger f'' is), the worse the approximation will be.

- SIMP(n) can be thought of as an attempt to approximate the graph of f using segments of parabolas.

The error behaves like $\frac{1}{n^4}$ and the bigger the fourth derivative $f^{(iv)}$ is, the worse the approximation will be.



- The better the desired approximation, the better it is to use Simpson's rule.

- We can also use higher degree polys. than quadratics as in Simpson. This is called quadrature.