Other uses for Taylor polynomials/series

MTH 142

While in class we typically compute derivatives of a given function and use those derivatives to find Taylor polynomials and series, we can also use the known terms of a Taylor series to evaluate higher derivatives of a function. For instance, if we define

In[157]:=
```
f[x_] := Exp[Cos[x]]
f[x]
```

Out[158]= $e^{\text{Cos}[x]}$

and need to compute several derivatives, and use the code to generate the nth Taylor polynomial about a

In[159]:=
```
P[n_, f_, a_, x_] := Normal[Series[f[t], {t, a, n}]] /. t → x
```

then just as in a previous assignment, we can have *Mathematica* generate any Taylor polynomial we wish. For instance, the 6th Taylor polynomial for f(x) about a=0 is

In[160]:=
```
P[6, f, 0, x]
```

Out[160]= $e - \frac{e x^2}{2} + \frac{e x^4}{6} - \frac{31 e x^6}{720}$

From the formula defining the Taylor polynomials, you may notice that the coefficient of the kth power of x is the kth derivative of f, evaluated at x=a, divided by k factorial. Thus, the 6th derivative of f evaluated at x=0 is

In[161]:=
```
-31 e / 720 * 6 !
```

Out[161]= $-31 e$

We can check that this is correct by asking *Mathematica* to take the 6th derivative for us,

In[162]:=
```
D[f[x], {x, 6}]
```

Out[162]= $-e^{\text{Cos}[x]} \text{Cos}[x] - 15 e^{\text{Cos}[x]} \text{Cos}[x]^2 - 15 e^{\text{Cos}[x]} \text{Cos}[x]^3 + 16 e^{\text{Cos}[x]} \text{Sin}[x]^2 + 75 e^{\text{Cos}[x]} \text{Cos}[x] \text{Sin}[x]^2 + 45 e^{\text{Cos}[x]} \text{Cos}[x]^2 \text{Sin}[x]^2 - 20 e^{\text{Cos}[x]} \text{Sin}[x]^4 - 15 e^{\text{Cos}[x]} \text{Cos}[x] \text{Sin}[x]^4 + e^{\text{Cos}[x]} \text{Sin}[x]^6$

and then asking it to evaluate this for us when x=0,

In[163]:=
```
D[f[x], {x, 6}] /. x → 0
```

Out[163]= $-31 e$

and we see these are equivalent.

Another use of Taylor series is in evaluating limits. Often L'Hopital's rule is difficult to apply multiple times, and we may instead replace the function with its Taylor series, and evaluate the limit in that way. Note that for the functions in this example, finding the Taylor series without using *Mathematica* would require starting with the known Taylor series for cos(x) and modifying it rather than computing the derivatives, which do not exist at x=0.

For instance, consider the function

In[164]:=
```
g[x_] := (x^2/2 - 1 + Cos[x]) / x^4
g[x]
```

Out[165]= $\dfrac{-1+\frac{x^2}{2}+\mathrm{Cos}[x]}{x^4}$

and its limit as x approaches zero. This is the indeterminant form 0/0, so we could apply L'Hopital's rule, but instead, let's look at some Taylor polynomials about x=0. The 10th Taylor polynomial about x=0 according to *Mathematica* is

In[166]:=
```
P[10, g, 0, x]
```

Out[166]= $\frac{1}{24}-\frac{x^2}{720}+\frac{x^4}{40320}-\frac{x^6}{3628800}+\frac{x^8}{479001600}-\frac{x^{10}}{87178291200}$

and we immediately notice that when x=0, this Taylor polynomial becomes simply the constant term

In[167]:=
```
P[10, g, 0, x] /. x → 0
```

Out[167]= $\frac{1}{24}$

From this, we conclude that the limit of f(x) as x approaches 0 is 1/24. We may check this with *Mathematica* directly using

In[168]:=
```
Limit[g[x], x → 0]
```

Out[168]= $\frac{1}{24}$

PROBLEMS

Start a new notebook, and begin with the commands:

```
f[x_] := (x^2 + 1) / (1 + x)^(1/3)
g[x_] := (2 Sin[x] - ArcTan[x] - x) / (2 x^5)
P[n_, f_, a_, x_] := Normal[Series[f[t], {t, a, n}]] /. t → x
```

Problem 1. By looking at Taylor polynomial approximations of f(x) about 0, compute the 7th derivative of f(x), evaluated at zero. Check your answer by having *Mathematica* directly perform the differentiation.

Problem 2. Use the Taylor polynomial approximations of g(x) about 0 to compute the limit of g(x) as x approaches 0. Check your answer by having *Mathematica* evaluate the limit.