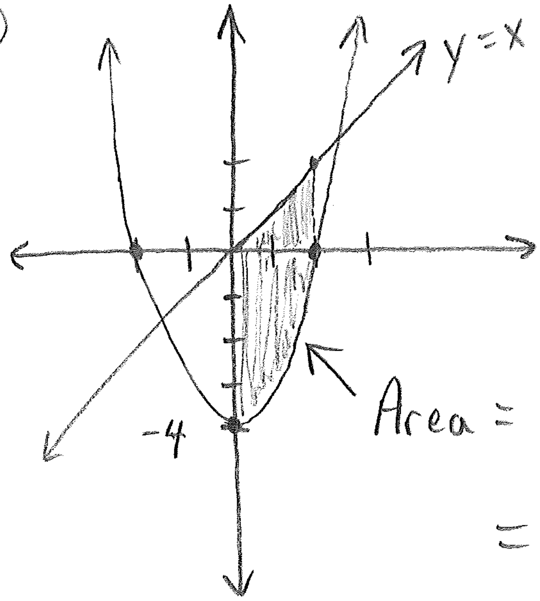
MTH 142 Worksheet #4

#1



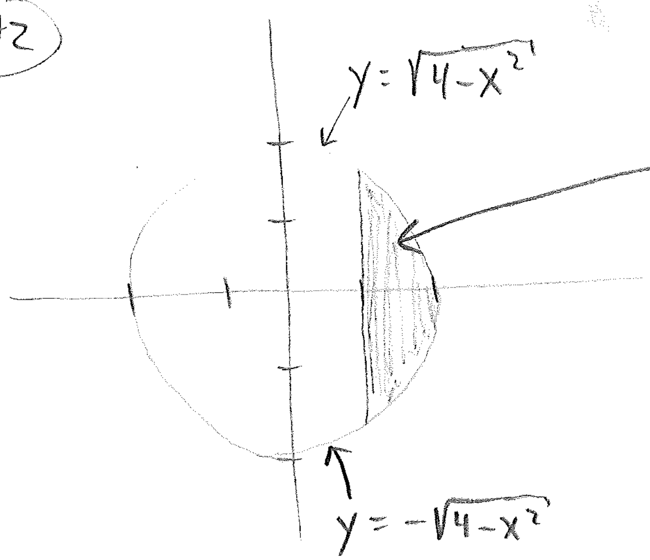
$$\text{Area} = \int_0^2 (x-(x^2-4))\,dx$$

$$= \int_0^2 (-x^2+x+4)\,dx$$

$$= -\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+4x\Big|_0^2$$

$$= -\tfrac{8}{3}+\tfrac{4}{2}+8 = 7\tfrac{1}{3}$$

#2



$$A = 2\cdot\int_1^2 \sqrt{4-x^2}\,dx$$

Let $x = 2\cdot\sin\theta$

$dx = 2\cdot\cos\theta\,d\theta$

So, $\int \sqrt{4-x^2}\,dx$

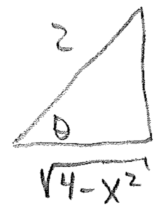
$$= 2\int \sqrt{4-4\sin^2\theta}\,\cos\theta\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta \quad \left(\text{note: } \tfrac{\pi}{6}\le\theta\le\tfrac{\pi}{2}\right)$$

$$= 4\cdot\left(\tfrac{1}{2}\cos\theta\cdot\sin\theta + \tfrac{1}{2}\int 1\,d\theta\right)$$

$$= 4\left(\tfrac{1}{2}\cos\theta \cdot \sin\theta + \tfrac{\theta}{2}\right) = 2\cos\theta\sin\theta + 2\theta$$

$\sin\theta = \dfrac{x}{2}$



So, $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

So, $\displaystyle\int \sqrt{4-x^2}\,dx = 2 \cdot \frac{\sqrt{4-x^2}}{2} \cdot \frac{x}{2} + 2 \cdot \sin^{-1}\left(\frac{x}{2}\right) + C$

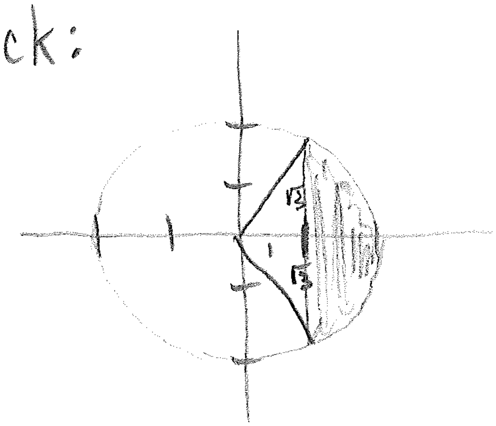
Thus, $A = 2 \cdot \displaystyle\int_1^2 \sqrt{4-x^2}\,dx$

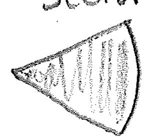
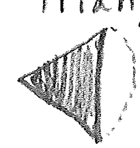
$$= 2 \cdot \left(\frac{x\sqrt{4-x^2}}{2} + 2 \cdot \sin^{-1}\left(\frac{x}{2}\right) \Bigg|_1^2 \right)$$

$$= 2 \cdot \left(\pi - \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3} \right) \right)$$

$$= 2 \cdot \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) = \frac{4\pi}{3} - \sqrt{3}$$

Check:

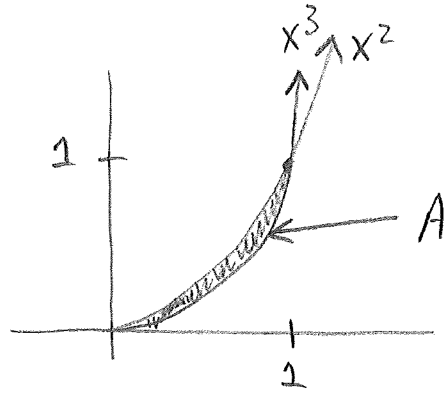


Area = sector  − triangle 

$$= \frac{1}{2} \cdot 2^2 \cdot \frac{2\pi}{3} - \sqrt{3}$$

$$= \frac{4\pi}{3} - \sqrt{3} \quad \checkmark$$

#3

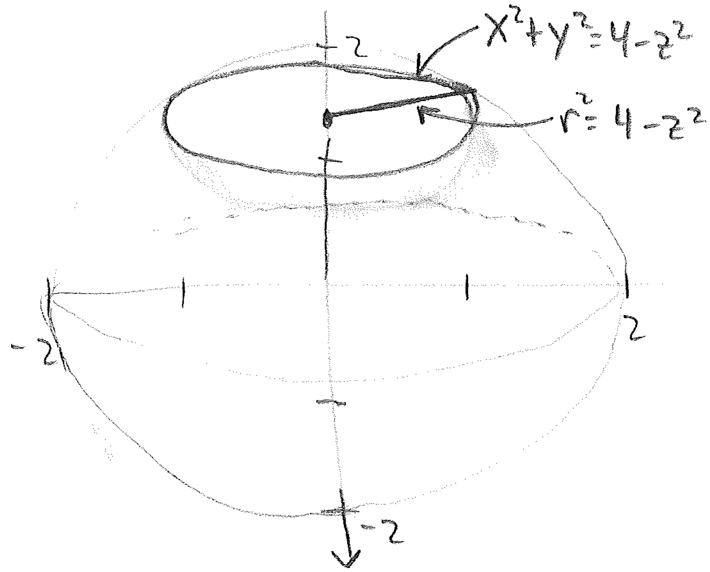


$$A = \int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

#4

$x^2 + y^2 + z^2 = 2^2$, so $x^2 + y^2 = 4 - z^2$



$$V = \int_1^2 \pi(4 - z^2)\,dz$$

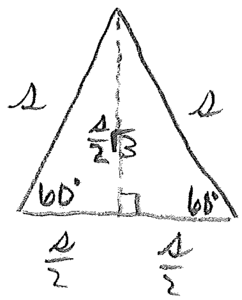
$$= \int_1^2 (4\pi - \pi z^2)\,dz$$

$$= 4\pi z - \frac{1}{3}\pi z^3 \Big|_1^2$$

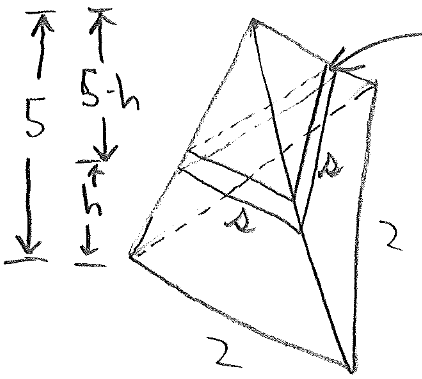
$$= 8\pi - \frac{8\pi}{3} - \left(4\pi - \frac{1}{3}\pi\right)$$

$$= 4\pi - \frac{7\pi}{3} = \frac{5\pi}{3}$$

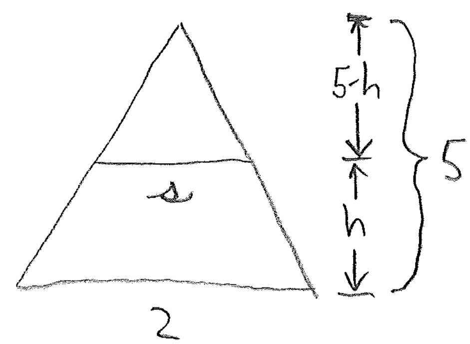
(#5) Area of an equilateral Δ given side s.



$$A_\Delta = \frac{1}{2}\cdot b\cdot h = \frac{1}{2}s\cdot\frac{s}{2}\sqrt{3} = \frac{s^2\sqrt{3}}{4}$$



Each slice is an equilateral Δ



$$\frac{s}{2} = \frac{5-h}{5} \Rightarrow s = \frac{2(5-h)}{5}$$

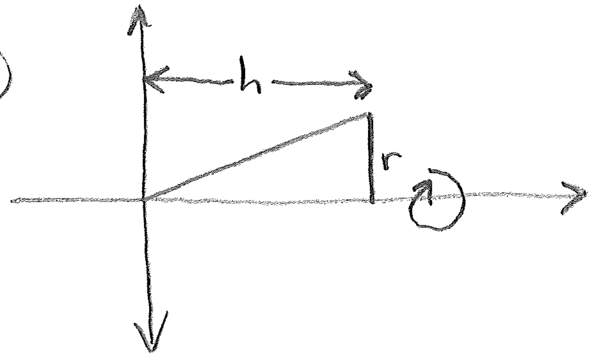
$$\text{Area}_\Delta = \frac{s^2\sqrt{3}}{4} = \frac{\left[\frac{2}{5}(5-h)\right]^2\sqrt{3}}{4} = \frac{\sqrt{3}}{25}(5-h)^2$$

$$\text{Vol} = \frac{\sqrt{3}}{25}\int_0^5 (25-10h+h^2)\,dh$$

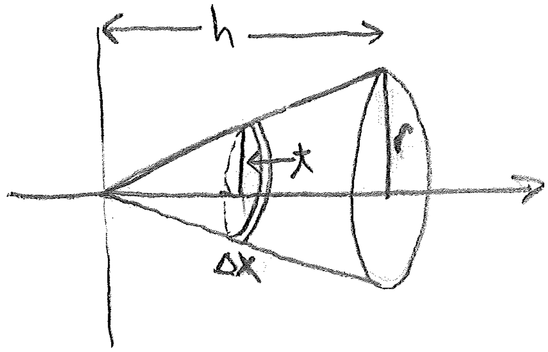
$$= \frac{\sqrt{3}}{25}\left(25h - 5h^2 + \frac{1}{3}h^3\Big|_0^5\right)$$

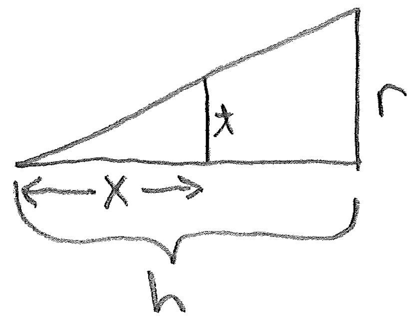
$$= \frac{\sqrt{3}}{25}\cdot\left(\cancel{125} - \cancel{125} + \frac{125}{3}\right) = \frac{5\sqrt{3}}{3} = \frac{1}{3}\cdot A_{base}\cdot \text{height}$$

#6



rotate the Δ around the x-axis





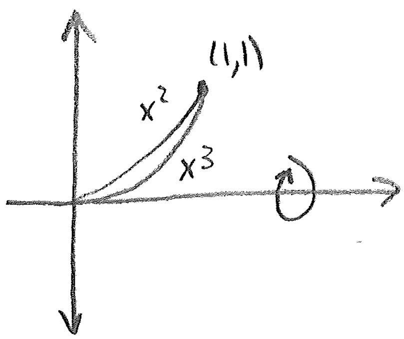
$$\frac{t}{r} = \frac{x}{h} \Rightarrow t = \frac{r}{h}\cdot x$$

r, h are constants

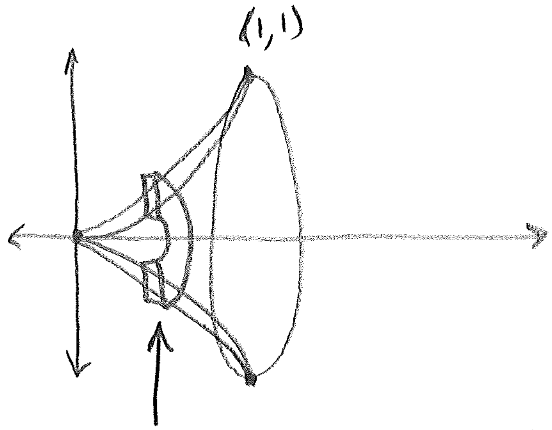
$$\text{Vol} = \int_0^h \pi\left(\frac{r}{h}x\right)^2 dx$$

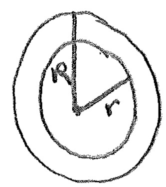
$$= \frac{\pi r^2}{h^2}\cdot\int_0^h x^2\,dx = \frac{\pi r^2}{h^2}\cdot\left(\frac{1}{3}x^3\Big|_0^h\right) = \frac{\pi r^2 h^3}{3h^2} = \frac{1}{3}\pi r^2 h$$

#7



Rotate around the x-axis.

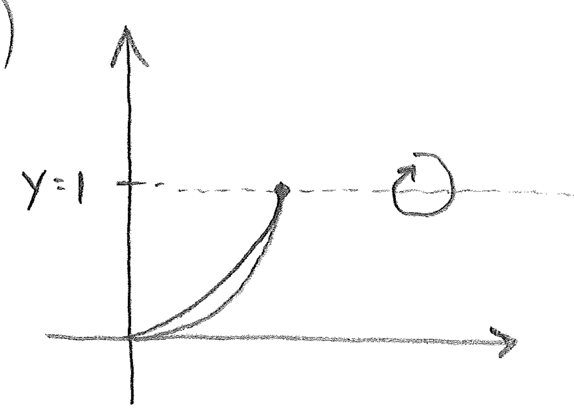


Each cross section looks like 

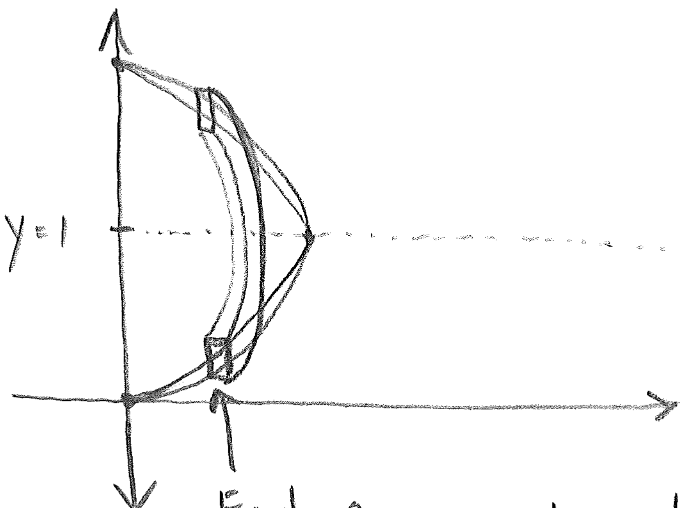
$$A = \pi R^2 - \pi r^2$$
$$= \pi(R^2 - r^2)$$
$$= \pi((x^2)^2 - (x^3)^2)$$
$$= \pi(x^4 - x^6)$$

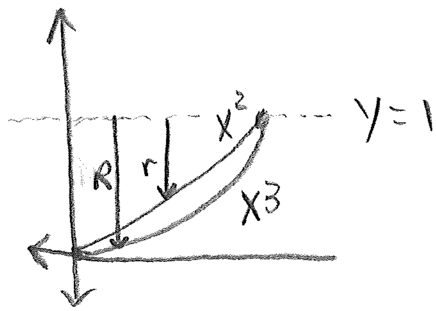
$$\text{Volume} = \pi \int_0^1 (x^4 - x^6)\,dx$$
$$= \pi\left(\tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 \Big|_0^1\right)$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{1}{7}\right) = \frac{2\pi}{35}$$

#7 (b)

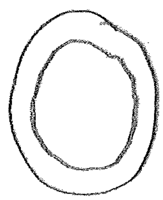


Rotate around line $y=1$





Each cross section looks like

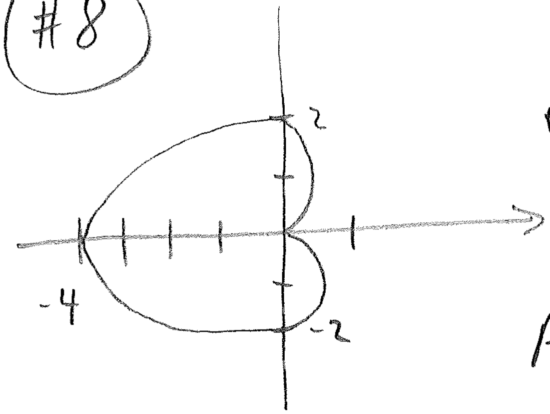


$$A = \pi(R^2 - r^2)$$
$$= \pi\left((1-x^3)^2 - (1-x^2)^2\right)$$
$$= \pi(x^6 - x^4 - 2x^3 + 2x^2)$$

$$\text{Vol} = \pi\int_0^1 (x^6 - x^4 - 2x^3 + 2x^2)\,dx$$

$$= \frac{23\pi}{210}$$

#8



$$r = 2(1-\cos\theta)$$

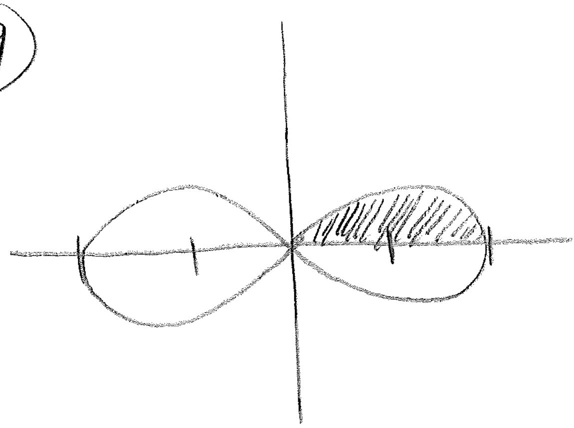
$$A = \int_0^{2\pi} \tfrac{1}{2} r^2 \, d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} (2(1-\cos\theta))^2 \, d\theta$$

$$= 2\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= 6\pi$$

#9



$$r^2 = 4\cos(2\theta)$$

A = shaded Area × 4

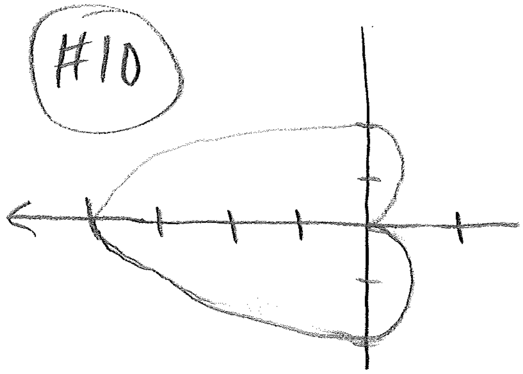
$$A = 4 \cdot \int_0^{\pi/4} \tfrac{1}{2} r^2 \, d\theta$$

$$= 2 \cdot \int_0^{\pi/4} (4\cos 2\theta) \, d\theta$$

$$= 8 \cdot \int_0^{\pi/4} \cos(2\theta) \, d\theta$$

$$= 4$$

#10



$$r = 2(1-\cos\theta) = 2 - 2\cos\theta$$

$$x = r\cos\theta = (2-2\cos\theta)\cdot\cos\theta = 2\cos\theta - 2\cos^2\theta$$

$$\frac{dx}{d\theta} = -2\sin\theta + 4\cos\theta\cdot\sin\theta$$

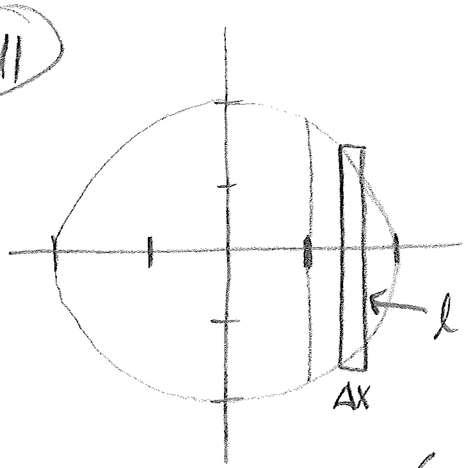
$$y = r\sin\theta = (2-2\cos\theta)\cdot\sin\theta = 2\sin\theta - 2\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = 2\cos\theta - 2(-\sin^2\theta + \cos^2\theta)$$

$$= 2\cos\theta + 2\sin^2\theta - 2\cos^2\theta$$

$$\text{Arc length} = \int_0^{2\pi}\sqrt{(-2\sin\theta + 4\sin\theta\cos\theta)^2 + (2\cos\theta + 2\sin^2\theta - 2\cos^2\theta)^2}\,d\theta$$

#11



$\ell = \sqrt{4-x^2} - (-\sqrt{4-x^2}) = 2\sqrt{4-x^2}$

So $A_x = 2\sqrt{4-x^2} \cdot \Delta x$

mass $= m$ $\quad \delta = \dfrac{\text{mass}}{\text{Area}} = \dfrac{m}{\frac{4\pi}{3} - \sqrt{3}}$

$$\bar{X} = \frac{\int_a^b x \cdot \delta \cdot A_x \, dx}{\text{mass}} = \frac{\int_1^2 x \cdot \frac{m}{\frac{4\pi}{3} - \sqrt{3}} \cdot 2\sqrt{4-x^2} \, dx}{m}$$

$$= \frac{2}{\frac{4\pi}{3} - \sqrt{3}} \cdot \int_1^2 x \cdot \sqrt{4-x^2} \, dx$$

Let $u = 4-x^2$, $du = -2x\,dx$, $-\frac{1}{2}du = x\,dx$

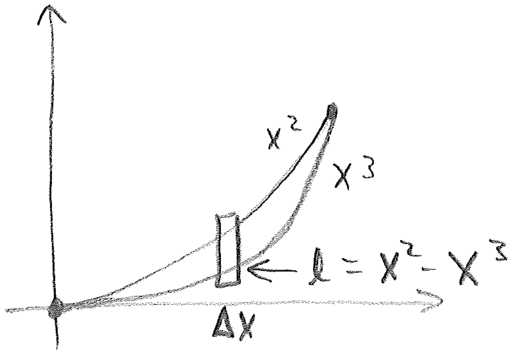
So, $\int x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2}$

$= -\frac{1}{3}(4-x^2)^{3/2}$

Thus, $\bar{X} = \dfrac{2}{\frac{4\pi}{3} - \sqrt{3}} \cdot \int_1^2 x\sqrt{4-x^2}\,dx = \dfrac{2}{\frac{4\pi}{3} - \sqrt{3}} \cdot \left(-\frac{1}{3}(4-x^2)^{3/2} \Big|_1^2\right)$

$= \dfrac{2}{\frac{4\pi}{3} - \sqrt{3}} \cdot \left(\frac{1}{3} \cdot 3^{3/2}\right) \approx 1.410$

#12



$A_x = (x^2 - x^3)\cdot\Delta x$

$\delta(x) = 2 - x$

For each slice that is $\perp$ to the x-axis, the density is constant (so long as the slices are skinny enough).

For each slice, $\delta_x = \dfrac{m_x}{A_x}$. So, $m_x = \delta_x \cdot A_x$

Thus, mass $= \int_0^1 \delta_x \cdot A_x \, dx = \int_0^1 (2-x)(x^2-x^3)\, dx$

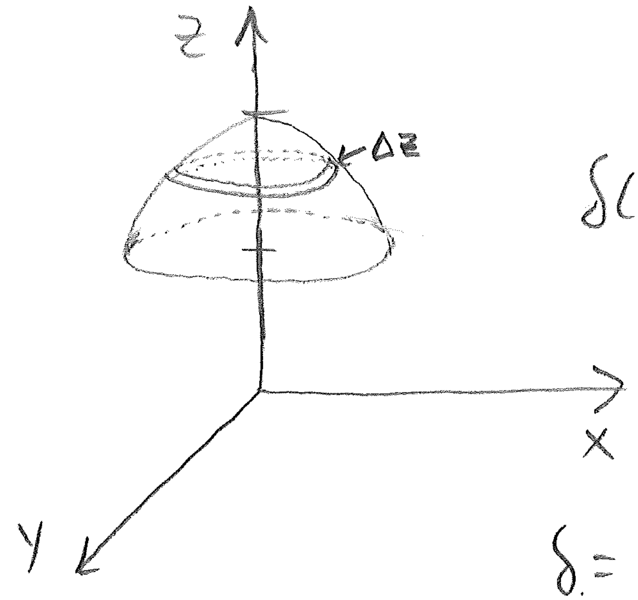
Since the density varies only with respect to x and the density for each A_x is constant, we can use the formula at the bottom of p. 395 for $\bar{x}$.

$$\therefore \; \bar{x} = \frac{\int_0^1 x\cdot(2-x)\cdot(x^2-x^3)\,dx}{\int_0^1 (2-x)\cdot(x^2-x^3)\,dx}$$

$$= \frac{\int_0^1 (x^5 - 3x^4 + 2x^3)\,dx}{\int_0^1 (x^4 - 3x^3 + 2x^2)\,dx} = \frac{4}{7}$$

Since the density is <u>not</u> constant for each slice $\perp$ to the y-axis, we may <u>not</u> use the formula at the bottom of p. 395 for $\bar{y}$.

#13



$$\delta(z) = 2 - \frac{1}{z}$$

$$\delta = \frac{\text{mass}}{\text{vol}}$$

$$m_z = \delta_z V_z$$

$$m_z = \left(2 - \frac{1}{z}\right) \cdot \left(\pi(4 - z^2)\right) \cdot \Delta z$$

$$\bar{z} = \frac{\int_1^2 z \cdot \left(2 - \frac{1}{z}\right) \cdot \pi \cdot (4 - z^2)\, dz}{\int_1^2 \left(2 - \frac{1}{z}\right) \cdot \pi \cdot (4 - z^2)\, dz}$$

$$\bar{z} \approx 1.375$$