WORKSHEET #3

#1 (i) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ $\quad \frac{1}{\sqrt{x}}$ is undefined if $x=0$

$$\lim_{n\to 0^+} \int_n^1 \frac{1}{\sqrt{x}}\,dx = \lim_{n\to 0^+} -\frac{1}{2}x^{-3/2}\Big|_n^1$$

$$= \lim_{n\to 0^+} \left(-\frac{1}{2} - \frac{-1}{2n^{3/2}}\right) = +\infty$$

So $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ does not converge.

(ii) $\int_{-\infty}^{-1} e^{2x}\,dx$

$$\lim_{n\to -\infty} \int_n^{-1} e^{2x}\,dx = \lim_{n\to -\infty} \frac{1}{2}e^{2x}\Big|_n^{-1}$$

$$= \lim_{n\to -\infty} \left(\frac{1}{2e^2} - \frac{e^{2n}}{2}\right)$$

$$= \frac{1}{2e^2}$$

So, $\int_{-\infty}^{-1} e^{2x}\,dx = \frac{1}{2e^2}$

(iii) $\int_0^\infty \frac{1}{x}\,dx$ $\quad \frac{1}{x}$ is undefined if $x=0$

$$\lim_{m\to 0^+}\left(\lim_{n\to\infty} \int_m^n \frac{1}{x}\,dx\right) = \lim_{m\to 0^+}\left(\lim_{n\to\infty} \ln x\Big|_m^n\right)$$

$$= \lim_{m\to 0^+}\left(\lim_{n\to\infty} (\ln n - \ln m)\right)$$

$$= \lim_{m\to 0^+} (\infty - \ln m) = \infty - (-\infty) = \infty$$

(iv) $\int_{-1}^{1} \ln|x|\,dx$ $\quad$ $\ln|x|$ is undefined if $x=0$

$$\int_{-1}^{1} \ln|x|\,dx = \int_{-1}^{0} \ln|x|\,dx + \int_{0}^{1} \ln|x|\,dx$$

($\ln|x|$ is continuous everywhere except at zero)

$$= \int_{-1}^{0} \ln(-x)\,dx + \int_{0}^{1} \ln x\,dx$$

$$\lim_{n\to 0^-} \int_{-1}^{n} \ln(-x)\,dx + \lim_{m\to 0^+} \int_{m}^{1} \ln x\,dx \quad \text{(use I.B.P.)}$$

$$= \lim_{n\to 0^-} x\cdot\ln(-x) - x \Big|_{-1}^{n} + \lim_{m\to 0^+} x\ln x - x \Big|_{m}^{1}$$

Note: $\lim_{n\to 0^-} n\cdot\ln(-n) = \lim_{n\to 0^-} \dfrac{\ln(-n)}{\frac{1}{n}} \overset{L.R.}{=} \lim_{n\to 0^-} \dfrac{-\frac{1}{-n}}{-\frac{1}{n^2}}$

$$= \lim_{n\to 0^-} -n = 0$$

Similarly, $\lim_{m\to 0^+} m\cdot\ln m = 0$

$$= (0-0)-(0+1)+(0-1)-(0-0) = -2$$

(v) $\int_{0}^{1} \dfrac{e^{-1/x}}{x^2}\,dx$ $\quad$ let $u = -\frac{1}{x}$ $\quad du = \frac{1}{x^2}\,dx$

as $x\to 0^+$, $u\to -\infty$ $\quad$ if $x=1$, $u=-1$

$$\int_{-\infty}^{-1} e^{u}\,du. \quad \lim_{n\to -\infty} \int_{n}^{-1} e^{u}\,du = \lim_{n\to -\infty} e^{u}\Big|_{n}^{-1} = e^{-1}$$

So, $\int_{0}^{1} \dfrac{e^{-1/x}}{x^2}\,dx = \dfrac{1}{e}$

#2 (i) $\int_0^\infty e^{-x}(1+\cos(x^2))\, dx$

$-1 \leq \cos(x^2) \leq 1$

So, $0 \leq 1+\cos(x^2) \leq 2$

$\therefore\ e^{-x}(1+\cos(x^2)) \leq 2e^{-x}$ for all $x \in \mathbb{R}$

$\lim_{n\to\infty} \int_0^n 2e^{-x}\, dx = \lim_{n\to\infty} -2e^{-x}\Big|_0^n = (0 - {-2}) = 2$

So, $\int_0^\infty e^{-x}(1+\cos(x^2))\, dx \leq 2$

Thus, the integral converges

(iii) $\int_0^\infty e^{-x^4}\, dx$

$e^x > x$ for all $x \in [0,\infty)$

So $e^{x^4} > x^4$ for all $x \in [0,\infty)$

Thus, $\frac{1}{e^{x^4}} < \frac{1}{x^4}$ for all $x \in (0,\infty)$ (note: $x \neq 0$)

So $\int_0^\infty \frac{1}{e^{x^4}}\, dx \leq \int_0^\infty \frac{1}{x^4}\, dx$ which converges by p-test (see top of pg. 358)

Thus, $\int_0^\infty \frac{1}{e^{x^4}}\, dx$ converges

(ii) $\int_0^1 \frac{2+\sin\left(\frac{1}{x}\right)}{x}\,dx$ $\quad$ $\frac{2+\sin\left(\frac{1}{x}\right)}{x}$ is undefined if $x=0$

$$-1 \le \sin\left(\tfrac{1}{x}\right) \le 1$$

$$1 \le 2+\sin\left(\tfrac{1}{x}\right) \le 3$$

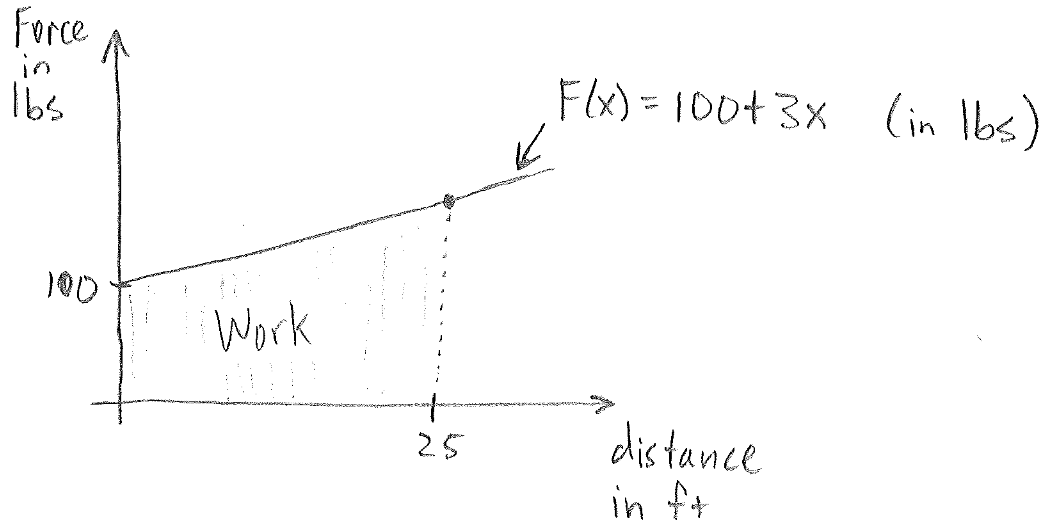
$$0 < \frac{1}{x} \le \frac{2+\sin\left(\frac{1}{x}\right)}{x} \le \frac{3}{x} \quad \text{for } x \in (0,1]$$

Thus, $\int_0^1 \frac{2+\sin\left(\frac{1}{x}\right)}{x}\,dx \ge \int_0^1 \frac{1}{x}\,dx$

However, $\int_0^1 \frac{1}{x}\,dx$ diverges, So $\int_0^1 \frac{2+\sin\left(\frac{1}{x}\right)}{x}\,dx$ diverges

Work sheet #5

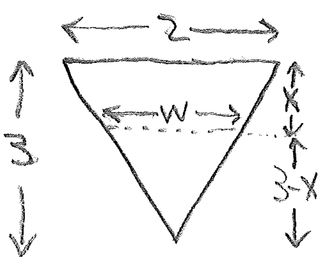
#1



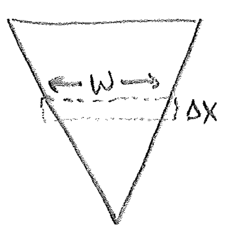
$$\text{Work} = \int_0^{25} F(x)\,dx = \int_0^{25} (100+3x)\,dx$$

$$= 100x + \tfrac{3}{2}x^2 \Big|_0^{25} = 3437.5 \text{ ft}\cdot\text{lbs}$$

#2 $\delta = 64.2 \text{ lbs}/\text{ft}^3$



So, $\dfrac{3-x}{3} = \dfrac{w}{2} \Rightarrow w = \tfrac{2}{3}(3-x) = 2 - \tfrac{2}{3}x$



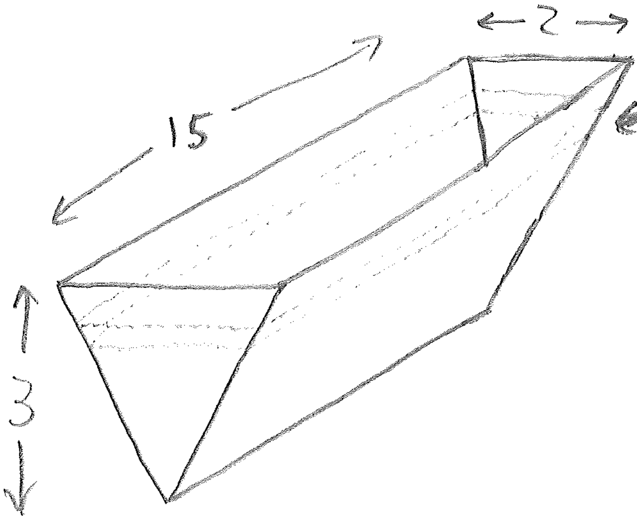
$\text{Area}_{\text{strip}} = (2 - \tfrac{2}{3}x)\cdot\Delta x \ \text{ft}^2$

$\text{Pressure}_{\text{strip}} = \delta\cdot h = 64.2x \ \text{lbs}/\text{ft}^2$

$\text{Force}_{\text{strip}} = P\cdot A_{\text{strip}} = 64.2x\cdot(2 - \tfrac{2}{3}x)\cdot\Delta x$

$= (128.4x - 42.8x^2)\cdot\Delta x \ \text{lbs}$

(a) $\text{Total Force} = \int_0^3 (128.4x - 42.8x^2)\,dx = 192.6 \text{ lbs}$



$$\text{Volume}_{\text{slice}} = \left(2 - \tfrac{2}{3}x\right) \cdot 15 \cdot \Delta x$$
$$= (30 - 10x) \cdot \Delta x \ \ ft^3$$

$$\text{Force}_{\text{slice}} = \delta \cdot V$$
$$= 64.2 \cdot (30 - 10x) \cdot \Delta x$$
$$= (1926 - 642x) \cdot \Delta x \ \ \text{lbs}$$

(b) $$\text{Work} = \int_0^3 \underbrace{(1926 - 642x) \cdot x}_{\text{Force} \times \text{Distance}} \, dx = 2889 \ ft \cdot lbs$$

Worksheet #6

#1.

$$p(x) = \begin{cases} c\cdot(2x-x^2) & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$p(x) \ge 0 \ \forall x.$

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^2 p(x)\,dx = c\cdot\int_0^2 (2x-x^2)\,dx$$

$$= c\cdot\left(x^2 - \tfrac{1}{3}x^3\Big|_0^2\right) = c\cdot\left(4-\tfrac{8}{3}\right) = \tfrac{4}{3}c$$

(i) If $\frac{4}{3}c = 1$, then $c = 3/4$.

(ii) $$P(t) = \int_{-\infty}^{t} \tfrac{3}{4}(2x-x^2)\,dx = \tfrac{3}{4}\int_0^t (2x-x^2)\,dx$$

$$= \tfrac{3}{4}\cdot\left(2t-t^2\right) = \tfrac{3}{2}t - \tfrac{3}{4}t^2$$

#2

$$P(t) = \begin{cases} 0 & t<0 \\ 1-e^{-t} & t \ge 0 \end{cases}$$

(i) $\lim_{t\to-\infty} P(t) = \lim_{t\to-\infty} 0 = 0$

$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} 1-e^{-t} = 1-0 = 1$

(ii) $$p(t) = P'(t) = \begin{cases} 0 & \text{if } t<0 \\ e^{-t} & \text{if } t \ge 0 \end{cases}$$

#3 (a) $c \cdot \int_0^6 e^{-ct}\, dt = 0.1$

$$c \cdot \left(-\frac{1}{c} \cdot e^{-ct} \Big|_0^6 \right) = 0.1$$

$$c \cdot \left(\frac{-e^{-6c}}{c} - \frac{-1}{c} \right) = 0.1$$

$$-e^{-6c} + 1 = 0.1$$

$$0.9 = e^{-6c}$$

$$\ln 0.9 = -6c$$

$$c = \frac{\ln(0.9)}{-6} \approx 0.0176$$

(b) $\frac{\ln(0.9)}{-6} \cdot \int_6^{12} e^{\frac{\ln(.9)}{-6} \cdot t}\, dt$

$$= \frac{\ln(0.9)}{-6} \cdot \left(\frac{-6}{\ln(.9)} e^{\frac{\ln(.9)}{-6} t} \Big|_6^{12} \right)$$

$$= e^{-2 \cdot \ln(.9)} - e^{-\ln(.9)} \approx 0.123456790\,1$$

#4

$$p(x) = \begin{cases} \frac{x^3}{4} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) $\int_{1.5}^{2} \frac{x^3}{4}\, dx = \frac{1}{16}x^4 \Big|_{1.5}^{2} = 1 - \frac{(1.5)^4}{16} \approx 0.684$ or 68.4%

(b) $\overline{X} = \int_{-\infty}^{\infty} x \cdot \frac{x^3}{4}\, dx = \int_0^2 \frac{x^4}{4}\, dx$

$= \frac{1}{20} \cdot x^5 \Big|_0^2 = \frac{8}{5} = 1.6$ hours

(c) $\int_{-\infty}^{m} \frac{x^3}{4}\, dx = 0.5$

$\frac{1}{16}x^4 \Big|_0^m = 0.5$

$\frac{m^4}{16} = 0.5$

$m^4 = 8$

median $= \sqrt[4]{8} \approx 1.682$ hours