Worksheet #8

1. **(a)** $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n^2+1}}{2n+1}$.

$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{2n+1} = \frac{1}{2} \neq 0$, so the series $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n^2+1}}{2n+1}$ diverges.

(b) $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-n}$.

$\frac{n^2+1}{n^3-n} > \frac{n^2}{n^3} = \frac{1}{n}$ for all $n \geq 2$. Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

by the Comparison Test, $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-n}$ also diverges.

(c) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^4+2n^2}$.

$\frac{n^2-1}{n^4+2n^2} < \frac{n^2}{n^4} = \frac{1}{n^2}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p > 1$),

the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^4+2n^2}$ converges by the Comparison Test.

Since $\frac{n^2-1}{n^4+2n^2} = \left|\frac{n^2-1}{n^4+2n^2}\right|$ for all $n \geq 1$,

the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^4+2n^2}$ converges absolutely.

(d) $\sum_{n=2}^{\infty} \frac{n^2+1}{n^4-n^2-1}$.

Let $a_n = \frac{1}{n^2}$ and let $b_n = \frac{n^2+1}{n^4-n^2-1}$.

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{1}{n^2} \div \frac{n^2+1}{n^4-n^2-1}\right) = \lim_{n\to\infty} \frac{n^4-n^2-1}{n^4+n^2} = 1$.

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges since it's a p-series.

By the Limit Comparison Test, $\sum_{n=2}^{\infty} \frac{n^2+1}{n^4-n^2-1}$ also converges.

Since $\frac{n^2+1}{n^4-n^2-1} = \left|\frac{n^2+1}{n^4-n^2-1}\right|$ for all $n \geq 2$,

the series $\sum_{n=2}^{\infty} \frac{n^2+1}{n^4-n^2-1}$ converges absolutely.

(e) $\sum_{n=1}^{\infty} \frac{2^n}{3^n \cdot n!}$.

$\frac{2^n}{3^n \cdot n!} \le \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$ for all $n \ge 1$.

Since $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series,

the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n \cdot n!}$ converges by the Comparison Test.

Since $\frac{2^n}{3^n \cdot n!} = \left|\frac{2^n}{3^n \cdot n!}\right|$ for all $n \ge 1$, the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n \cdot n!}$ converges absolutely.

(f) $\sum_{n=1}^{\infty} \frac{3^{n^2}}{4^n}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{3^{(n+1)^2}}{4^{n+1}} \cdot \frac{4^n}{3^{n^2}} \right| = \lim_{n\to\infty} \left(\frac{1}{4} \cdot 3^{2n+1}\right) = \infty .$$

That is, $\frac{1}{4} \cdot 3^{2n+1}$ is strictly increasing and has no upper bound (exponential growth)

So, by the Ratio Test, $\sum_{n=1}^{\infty} \frac{3^{n^2}}{4^n}$ diverges.

(g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

$0 < \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$ for all $n \ge 1$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

So, by the Alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges.

$\left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{n^{\frac{1}{2}}}$. The series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges (p-series),

so the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ does ***not*** absolutely converge.

2. **(a)** $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n \cdot 3^n}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{(n+1)\cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{|x-2|^n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|x-2|}{3} = \frac{1}{3}|x-2|.$$

So, the radius of convergence is 3.
Check the endpoints of (–1,5):

$\sum_{n=0}^{\infty} \frac{(-1-2)^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ which converges (alternating harmonic series).

$\sum_{n=0}^{\infty} \frac{(5-2)^n}{n \cdot 3^n} = \sum_{n=0}^{\infty} \frac{1}{n}$ which diverges.

So the interval of convergence is [–1,5).

(b) $\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n^2 (x-0)^n$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|(n+1)^2 x^{n+1}\right|}{\left|n^2 x^n\right|} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot |x| = 1 \cdot |x-0|.$$

So, the radius of convergence is 1.
Check the endpoints of (–1,1):

$\sum_{n=0}^{\infty} n^2 (-1)^n$ diverges. $\sum_{n=0}^{\infty} n^2 (1)^n$ also diverges.

So the interval of convergence is (–1,1).

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|x^{2(n+1)}\right|}{(2(n+1))!} \cdot \frac{(2n)!}{\left|x^{2n}\right|} = \lim_{n\to\infty} \frac{(2n)(2n-1)(2n-2)\cdot \ldots \cdot 1}{(2n+2)(2n+1)(2n)(2n-1)\cdot \ldots \cdot 1} \cdot x^2 = \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+1)} = 0$$

So the radius of convergence is infinite.
Thus, the interval of convergence is $(-\infty, \infty)$.

for Dr. Comerford, MTH 142
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