Math 142 Worksheet # 11 - Solutions

1. i) $\int e^{\sin^2 t} \sin t \cos t \, dt$

Let $w = \sin^2 t$

$dw = 2 \sin t \cos t \, dt$

$\frac{dw}{2} = \sin t \cos t \, dt$

Get

$$\int e^{w} \cdot \frac{dw}{2} = \frac{1}{2} \int e^{w} dw$$

ii) $\int x^2 \sqrt{1+x^3} \, dx$

Let $w = 1 + x^3$

$dw = 3x^2 \, dx$

$\frac{dw}{3} = x^2 \, dx$

Get $\int \sqrt{w} \cdot \frac{dw}{3} = \frac{1}{3} \int \sqrt{w} \, dw$.

iii) $\displaystyle\int \frac{\sec^2(2x)}{\tan^7(2x)}\,dx$

Let $w = \tan(2x)$

$dw = 2\sec^2(2x)$

$\frac{dw}{2} = \sec^2(2x)$

Get $\displaystyle\int \frac{\frac{dw}{2}}{w^7} = \frac{1}{2}\int w^{-7}\,dw$

2. i) $x = 3\sin\theta \Rightarrow \sqrt{9-x^2}$

$= \sqrt{9-9\sin^2\theta}$

$= \sqrt{9(1-\sin^2\theta)}$

$= \sqrt{9}\sqrt{1-\sin^2\theta}$

$= 3\sqrt{\cos^2\theta}$ as $\sin^2\theta + \cos^2\theta = 1$

$= 3\cos\theta$

For $\int_0^3 \sqrt{9-x^2}\,dx$, if we let

$x = 3\sin\theta$, then when $x=0$, $\theta = 0$

$dx = 3\cos\theta$ $\qquad$ $x = 3$, $\theta = \frac{\pi}{2}$

$\left(3\sin\frac{\pi}{2} = 3(1) = 3\right)$.

and so

$$\int_0^3 \sqrt{9-x^2}\,dx = \int_0^{\frac{\pi}{2}} 3\cos\theta \cdot 3\cos\theta\, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 9\int_0^{\frac{\pi}{2}} \left[\frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= \frac{9}{2}\int_0^{\frac{\pi}{2}} (1+\cos(2\theta))\,d\theta$$

$$= \frac{9}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}} = \frac{9}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0\right]$$

$$= \frac{9\pi}{4}.$$

ii) If $x = 4\tan\theta$,

$$16 + x^2 = 16 + 16\tan^2\theta$$

$$= 16(1 + \tan^2\theta)$$

$$= 16\sec^2\theta \quad \text{as } \sec^2\theta = 1 + \tan^2\theta$$

For $\int_0^4 \frac{dx}{16+x^2}$, if we let $x = 4\tan\theta$,

$dx = 4\sec^2\theta\, d\theta$ and when $x = 0$, $\theta = 0$

$x = 4$, $\theta = \frac{\pi}{4}$

$(4\tan\frac{\pi}{4} = 4(1) = 4)$.

Thus

$$\int_0^4 \frac{dx}{16+x^2} = \int_0^{\frac{\pi}{4}} \frac{4\sec^2\theta\, d\theta}{16\sec^2\theta}$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{4}} d\theta = \frac{1}{4}\cdot\frac{\pi}{4} = \frac{\pi}{16}.$$

3.

$$\int e^{2x} \sin(3x)\,dx$$

By parts. $u = e^{2x}$ $dv = \sin(3x)\,dx$

$du = 2e^{2x}\,dx$ $v = -\frac{1}{3}\cos 3x$

$$= -\frac{1}{3} e^{2x} \cos(3x) - \int -\frac{1}{3}\cos(3x) \cdot 2e^{2x}\,dx.$$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3}\int e^{2x} \cos(3x)\,dx.$$

By parts again $u = e^{2x}$, $dv = \cos(3x)$

$du = 2e^{2x}\,dx$, $v = \frac{1}{3}\sin(3x)$.

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{3}\left\{ \frac{1}{3} e^{2x} \sin(3x) - \int \frac{1}{3}\sin(3x) \cdot 2e^{2x}\,dx \right\}$$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x) - \frac{4}{9}\int e^{2x} \sin(3x)\,dx$$

Thus

$$\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x)$$

$$-\frac{4}{9}\int e^{2x}\sin(3x)\,dx.$$

same integral as lhs.

and so

$$\frac{13}{9}\int e^{2x}\sin(3x)\,dx = -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) + C'$$

C' constant.

$$\int e^{2x}\sin(3x)\,dx = -\frac{3}{13}e^{2x}\cos(3x) + \frac{2}{13}e^{2x}\sin(3x) + C$$

$C = \frac{9}{13}C'$ constant.

4. $$\frac{1}{x^2-2x} = \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$= \frac{A(x-2)}{x(x-2)} + \frac{Bx}{x(x-2)}.$$

Thus

$$1 = A(x-2) + Bx$$

$$0x + 1 = (A+B)x - 2A.$$

Equating coeffts of x gives $\quad A+B=0 \Rightarrow B=-A$

Equating constant coeffts. gives $\quad -2A = 1 \Rightarrow A = -\frac{1}{2}$

$$\Rightarrow B = -A = \frac{1}{2}.$$

Thus

$$\frac{1}{x^2-2x} = -\frac{1}{2x} + \frac{1}{2(x-2)}.$$

5. Split up the interval of integration :

$$\int_0^\infty \frac{1}{x}\,dx = \int_0^1 \frac{1}{x}\,dx + \int_1^\infty \frac{1}{x}\,dx$$

Both these integrals are improper and we investigate their convergence separately.

$$\int_0^1 \frac{1}{x}\,dx = \lim_{a \to 0_+} \int_a^1 \frac{1}{x}\,dx = \lim_{a \to 0_+} \Big[\ln|x|\Big]_a^1$$

$$= \lim_{a \to 0_+} (\ln 1 - \ln a)$$

$$= \lim_{a \to 0_+} (-\ln a) = \infty$$

diverges.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\,dx = \lim_{b \to \infty} \Big[\ln|x|\Big]_1^b$$

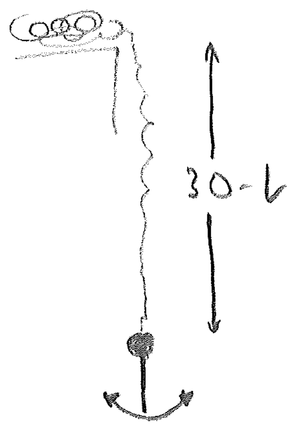
$$= \lim_{b \to \infty} (\ln b - \ln 1)$$

$$= \lim_{b \to \infty} \ln b = \infty$$

- diverges.

Thus $\int_0^\infty \frac{1}{x}dx$ diverges. Note that to show $\int_0^\infty \frac{1}{x}\,dx$ diverges, it would have been enough to show that just one of these integrals diverges.

6.



Suppose we have already hauled up L ft of chain so that $30-L$ feet are left.

The weight of the remaining chain + the anchor is

$$(30-L)(2)+20 = 80-2L \text{ lbs}$$

and the work required to raise this weight a small distance ΔL is then

$$(80-2L)\Delta L \text{ ft lbs.}$$

(N.b. we don't need to multiply by g ($=32 ft/s^2$) as we're using English units).

The total work is then approx.

$$W \approx \sum_{i=1}^{n} (80-2L)\Delta L$$

and as we let $\Delta L \to 0$ this Riemann sum becomes an integral and we get

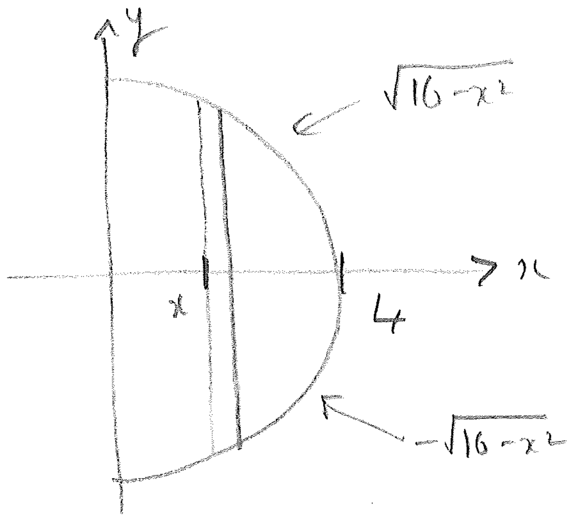
$$W = \int_0^{30} (80-2L)\,dL$$

$$= \left[80L - \frac{L^2}{2}\right]_0^{30}$$

$$= \left(2400 - \frac{900}{2} - (0-0)\right)$$

$= 1950$ ft lbs (n.b. don't forget to give the correct units in your answer!).

7.



Let σ be the (uniform) density of the region.

By symmetry $\bar{y} = 0$ and so we just need to find $\bar{x}$.

Slice the region up into vertical strips of width Δx as above.

Each strip is approx a rectangle of width Δx which lies between $\sqrt{16-x^2}$ and $-\sqrt{16-x^2}$ and so has height $2\sqrt{16-x^2}\,dx$.

The mass of this strip is then approx.

$$\sigma \cdot 2\sqrt{16-x^2} \cdot \Delta x. \quad = \sigma\, A_x(x)\, \Delta x.$$

Then

$$M_x = \int_0^4 x\,\sigma\,A_x(x)\,dx$$

$$= \int_0^4 \sigma 2x\sqrt{16-x^2}\,dx$$

$$= \sigma\int_0^4 2x\sqrt{16-x^2}\,dx.$$

let $w = 16-x^2$, $dw = -2x\,dx$.
$-dw = 2x\,dx$

when $x=0$ $w=16$
$x=4$ $w=0$

$$= \sigma\int_{16}^{0} \sqrt{w}\cdot -dw$$

$$= \sigma\int_0^{16} w^{\frac{1}{2}}\,dw.$$

$$= \sigma \left[\frac{2}{3} w^{3/2} \right]_0^{16}$$

$$= \sigma \left(\frac{2}{3} \cdot 64 - 0 \right)$$

$$= \frac{128\sigma}{3}.$$

On the other hand, since the density is <u>uniform</u>

$$M = \sigma\,(\text{Area})$$

$$= \sigma \cdot \frac{\pi (4)^2}{2}$$

$$= 8\pi\sigma$$

Then $\bar{x} = \dfrac{M_x}{M} = \dfrac{\frac{128\sigma}{3}}{8\pi\sigma} = \dfrac{16}{3\pi}$

Note that $0 < \frac{16}{3\pi} < 4$ and in fact

$0 < \frac{16}{3\pi} < 2$ as we'd expect (why?).

8. Need

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

ie $$\int_0^1 kx\,dx = 1$$

$$\left[k x^2/2 \right]_0^1 = 1$$

$$k\left(\tfrac{1}{2} - 0\right) = 1$$

$$\frac{k}{2} = 1$$

ie $k = 2$.

Thus $$p(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then for $0 < t < 1$,

$$P(t) = \int_0^t 2x\,dx = \left[x^2\right]_0^t = t^2$$ and so

the CDF is

$$P(t) = \begin{cases} t^2, & 0 < t < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

9. $$\sum_{n=1}^{\infty} \frac{n^2+n}{n^4-n^3+2}$$

For n large

$$\frac{n^2+\cancel{n}}{n^4-\cancel{n^3}+\cancel{2}} \approx \frac{n^2}{n^4} = \frac{1}{n^2},$$

So compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\frac{n^2+n}{n^4-n^3+2} \leq \frac{2n^2}{n^4-n^3+2} \leq \frac{2n^2}{n^4-n^3}$$

$$\leq \frac{2n^2}{n^4-n^4/2} \quad \text{for } n \geq 2$$

$$= \frac{2n^2}{n^4/2}$$

$$= \frac{4}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{4}{n^2}$ is convergent (p-series with $p=2>1$),

$\sum_{n=1}^{\infty} \frac{n^2+n}{n^4-n^3+2}$ is conv. by the comparison test.

Actually, it would be easier to do this particular example using the limit comparison test.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$$

for n large $\frac{1}{\sqrt{n^2 \cancel{-1}}} \approx \frac{1}{\sqrt{n^2}} = \frac{1}{n}$

so compare with $\frac{1}{n}$.

$$\frac{\frac{1}{\sqrt{n^2-1}}}{\frac{1}{n}} = \frac{n}{\sqrt{n^2-1}} = \frac{\sqrt{n^2}}{\sqrt{n^2-1}}$$

$$= \sqrt{\frac{n^2}{n^2-1}}$$

$$= \sqrt{\frac{1}{1-\frac{1}{n^2}}}$$

$$\longrightarrow 1 \text{ as } n \to \infty.$$

Since this limit exists and is > 0, it follows by the limit comparison test that since $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent (harmonic series),

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$ is also divergent.

10 $$\sum_{n=1}^{\infty} \frac{3^{n^2}}{(2n)!}$$

Here $$a_n = \frac{3^{n^2}}{(2n)!}$$

so $$a_{n+1} = \frac{3^{(n+1)^2}}{(2(n+1))!}$$

$$= \frac{3^{n^2+2n+1}}{(2n+2)!}$$

$$= \frac{3^{n^2+2n+1}}{(2n+2)(2n+1)(2n)!}$$

Thus $$\frac{a_{n+1}}{a_n} = \frac{3^{n^2+2n+1}}{(2n+2)(2n+1)(2n)!} \Big/ \frac{3^{n^2}}{(2n)!}$$

$$= \frac{3^{n^2+2n+1}/3^{n^2}}{(2n+2)(2n+1)(2n)!/(2n)!}$$

$$= \frac{3^{2n+1}}{(2n+2)(2n+1)}.$$

This $\to \infty$ as $n \to \infty$ (exponentials always grow faster than polynomials) and so the series diverges by the ratio test

11.

$$(1+2x)^{\frac{1}{3}} = 1 + \frac{1}{3}(2x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(2x)^2$$

$$+ \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{3!}(2x)^3 + \cdots$$

$$= 1 + \frac{2x}{3} - \frac{1}{9} \cdot 4x^2 + \frac{10}{9 \cdot 6} 8x^3 + \cdots$$

$$= 1 + \frac{2x}{3} - \frac{4x^2}{9} + \frac{40}{27}x^3 + \cdots$$

valid for $|2x| < 1$
ie $|x| < \frac{1}{2}$.

12. $g(x) = \sin(2x)$, $g(0) = \sin(0) = 0$

$g'(x) = 2\cos(2x)$, $g'(0) = 2\cos 0 = 2$

$g''(x) = -4\sin(2x)$, $g''(0) = -4\sin 0 = 0$

$g'''(x) = -8\cos(2x)$, $g'''(0) = -8\cos 0 = -8$

Thus

$$P_3(x) = g(0) + g'(0)x + \frac{g''(0)x^2}{2!} + \frac{g'''(0)x^3}{3!}$$

$$= 0 + 2x + 0x^2 - \frac{8x^3}{6}$$

$$= 2x - \frac{4x^3}{3}.$$

Now $g^{(4)}(x) = 16\sin(2x)$

and so $-16 \le g^{(4)}(x) \le 16$ on $[-1, 1]$

and we can take $M = 16$ in the Lagrange error bound.

Then

$$|E_3(x)| \le \frac{M}{4!}x^4$$

$$= \frac{16x^4}{24}$$

$$= \frac{2x^4}{3}, \qquad -1 \le x \le 1.$$