MTH 142 Worksheet #10

#1 (a) $\frac{dz}{dy} = zy \qquad z(0) = 1$

$$\int \frac{1}{z}\,dz = \int y\,dy$$

$$\ln|z| = \frac{1}{2}y^2 + C_1$$

$$z = \pm e^{\frac{1}{2}y^2 + C_1} = Ce^{\frac{1}{2}y^2}$$

$$1 = Ce^{\frac{1}{2}(0)^2} \Rightarrow C = 1$$

So $z = e^{\frac{1}{2}y^2}$

(b) $\frac{dy}{dt} = 4 + y^2 \qquad y(0) = 0$

$$\int \frac{dy}{4+y^2} = \int dt$$

$$\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = t + C_1$$

$$\frac{y}{2} = \tan(2t + C), \qquad C = 2C_1$$

$$y = 2\tan(2t + C)$$

$$0 = 2\tan(2\cdot 0 + C)$$

$$0 = \tan(C)$$

$$C = \tan^{-1}(0) = 0$$

So $y = 2\tan(2t)$

(c) $\frac{dw}{d\theta} = \theta w^2 \sin(\theta^2)$ $\quad w(0) = 1$

$$\int \frac{dw}{w^2} = \int \theta \sin(\theta^2)\, d\theta$$

substitution.
(Let $u = \theta^2$, $du = 2\theta\, d\theta$).

$$-w^{-1} = -\frac{1}{2}\cos(\theta^2) + C_1$$

$$\frac{1}{w} = \frac{1}{2}\cos(\theta^2) + C_2$$

$w(0)=1 \Rightarrow$ $\frac{1}{1} = \frac{1}{2}\cos(0^2) + C_2$

$$C_2 = \frac{1}{1} - \frac{1}{2}$$

$$\frac{1}{w} = \frac{1}{2}\cos(\theta^2) + \frac{1}{2}$$

$$w = \frac{2}{\cos(\theta^2) + 1}$$

(d) $x \cdot (x+1)\frac{du}{dx} = u^2$ $\quad u(1) = 1$

$$\int \frac{du}{u^2} = \int \frac{dx}{x(x+1)}$$

partial fractions $\frac{1}{x(x+1)}$
$= \frac{1}{x} - \frac{1}{x+1}$

$$-u^{-1} = \ln x - \ln(x+1) + C_1$$

$$\frac{1}{u} = \ln(x+1) - \ln x + C_2$$

$u(1)=1 \Rightarrow$ $1 = \ln 2 - \ln 1 + C_2$

$C_2 = -\ln 2$ $\quad$ So, $u = \frac{1}{\ln(x+1) - \ln(2x)}$

#2 (a) $\frac{dH}{dt} = -k(H-200)$ $\quad k>0$

$H = 20°C$ when $t=0$

$$\frac{dH}{H-200} = -k\,dt$$

$$\ln|H-200| = -kt + C_1$$

$$|H-200| = e^{-kt+C_1} = Ce^{-kt}$$

$$|20-200| = Ce^{-k(0)} \Rightarrow C = 180$$

$|H-200| = 200-H$ since $H \leq 200$ for all t

So, $H = 200 - 180e^{-kt}$

(b) when $t = 30$ sec, $H = 120°C$

$$120 = 200 - 180e^{-k(30)}$$

$$\frac{4}{9} = e^{-30k}$$

$$\ln\left(\frac{4}{9}\right) = -30k$$

$$k = \frac{\ln\left(\frac{4}{9}\right)}{-30}$$

(c) Find t when $H = 150$

$$150 = 200 - 180 e^{-\frac{\ln\left(\frac{4}{9}\right)}{-30} \cdot t}$$

$$\frac{5}{18} = e^{\frac{\ln 4/9}{30} t}$$

$$\frac{30 \cdot \ln\left(\frac{5}{18}\right)}{\ln\left(4/9\right)} = t \approx 47.4 \text{ s.}$$

#3 (a) $\frac{dP}{dt} = 2 \cdot P\left(1 - \frac{P}{200}\right)$ $\quad P = 20$ when $t = 0$

$$\int \frac{dP}{P\left(1 - \frac{P}{200}\right)} = \int 2\,dt$$

$$\ln|P| - \ln|P - 200| = 2t + C_1$$

$$-\ln\left|\frac{P-200}{P}\right| = -2t + C_2$$

$$-\left|\frac{P-200}{P}\right| = e^{-(2t+C_2)} = Ce^{-2t}$$

$$-\left|\frac{20-200}{20}\right| = Ce^{-2 \cdot 0} \Rightarrow C = -9$$

$$-\left|\frac{P-200}{P}\right| = -9e^{-2t}$$

Since $P \leq 200$, $\frac{200-P}{P} = 9e^{-2t}$

$$P = \frac{200}{1 + 9e^{-2t}}$$

#3 (b) 200 rabbits

(c) When $P = 100$

That is, $100 = \dfrac{200}{1+9e^{-2t}} \Rightarrow t = \ln 3 \approx 1.1$ months

#4 $f(x) = x$, $f(x+2) = f(x)$ for all x

$[-1,1] = [-b/2, b/2] \Rightarrow b = 2$

$$a_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = 0$$

$$a_k = \frac{2}{2}\int_{-1}^{1} x\cos(\pi k x)\,dx = 0$$

$$b_k = \frac{2}{2}\int_{-1}^{1} x\sin(\pi k x)\,dx$$

$$= \left.\frac{\sin(\pi k x)}{\pi^2 k^2} - \frac{x\cdot\cos(k\pi x)}{\pi k}\right|_{-1}^{1}$$

$$= \left(0 - \frac{\cos(k\pi)}{k\pi}\right) - \left(0 - \frac{(-1)\cdot\cos(k\pi)}{k\pi}\right)$$

$$= \frac{-2\cdot\cos(k\pi)}{k\pi} = \frac{-2\cdot(-1)^k}{k\pi}$$

$$f(x) = \sum_{k=1}^{\infty}\left(\frac{-2\cdot(-1)^k}{k\pi}\cdot\sin(k\pi x)\right)$$

#3 b)

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{200}{1+9e^{-2t}} = \frac{200}{1+9(0)} = 200$$

cute robbits.

c) The rabbits are breeding fastest when

$P = 100$.

At this time

$$100 = \frac{200}{1+9e^{-2t}}$$

$$\frac{1}{2} = \frac{1}{1+9e^{-2t}}$$

$$2 = 1+9e^{-2t}$$

$$9e^{-2t} = 1$$

$$e^{-2t} = \frac{1}{9}$$

$$-2t = \ln\left(\frac{1}{9}\right)$$

$$t = -\frac{1}{2}\ln\left(\frac{1}{9}\right) = \ln 3 \approx 1.1 \text{ months.}$$

#4 $f(x) = x$.

$[-1, 1] = [-b/2, b/2]$, so $b = 2$.

$$a_0 = \frac{1}{2}\int_{-1}^{1} x\,dx = 0$$ (odd fn on a symmetric interval).

$$a_k = \frac{2}{2}\int_{-1}^{1} x\cos(\pi k x)\,dx = 0$$ (again an odd fn on a symmetric interval).

$$b_k = \frac{2}{2}\int_{-1}^{1} x\sin(\pi k x)\,dx$$

$$= \int_{-1}^{1} x\sin(\pi k x)\,dx.$$ by parts.

$u = x$ $\quad dv = \sin(\pi k x)\,dx$

$du = dx$ $\quad v = -\cos(\pi k x)$

$$= \left[-x\frac{\cos(\pi k x)}{\pi k}\right]_{-1}^{1} - \int_{-1}^{1} -\frac{\cos(\pi k x)}{\pi k}\,dx$$

$$= \left[-x\frac{\cos(\pi k x)}{\pi k}\right]_{-1}^{1} + \int_{-1}^{1} \frac{\cos(\pi k x)}{\pi k}\,dx$$

$$= \left[\frac{-x\cos(\pi k x)}{\pi k} \right]_{-1}^{1} + \left[\frac{\sin(\pi k x)}{\pi^2 k^2} \right]_{-1}^{1}$$

$$= -1\frac{(-1)^k}{\pi k} - \left(-(-1)\cdot\frac{(-1)^k}{\pi k}\right) + \frac{0}{\pi^2 k^2} - \frac{0}{\pi^2 k^2}$$

using $\cos(\pi k x) = \cos(-\pi k x) = (-1)^k$

$\sin(\pi k x) = \sin(-\pi k x) = 0$

$$= -\frac{2}{\pi k}(-1)^k$$

$$= \frac{2}{\pi k}(-1)^{k+1}$$

Thus.

$$x = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{\pi k}\sin(\pi k x)$$