INTERLUDE

§ 4.8 A Little Bit about Parametric Curves

Consider a particle moving in the plane. Its position as a fn of time t is given by

$$x = f(t), \quad y = g(t)$$

or

$$(x, y) = (f(t), g(t))$$

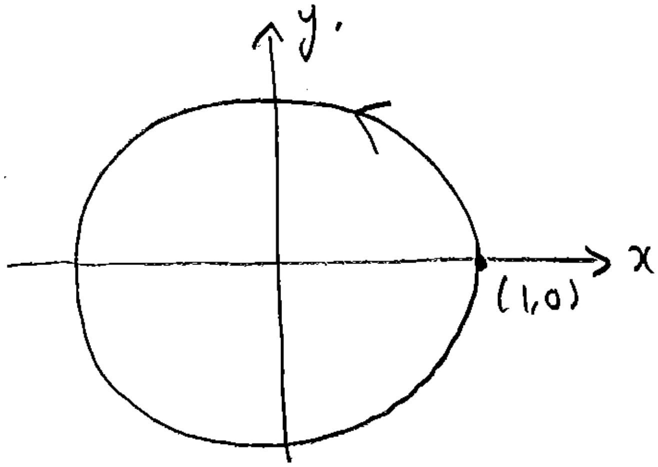
where f, g are fns of time t.

This is an example of a parametric curve.

Ex. $x = \cos t, \; y = \sin t, \quad 0 \leq t \leq 2\pi.$

Here $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$
and so we are on the unit circle.

As t runs from 0 to 2π we go once around the circle anticlockwise, starting and finishing at the pt $(\cos 0, \sin 0) = (\cos 2\pi, \sin 2\pi) = (1, 0)$



Ex: A straight line.

Here x, y change at a constant rate in t, so

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b \quad \text{for some constants } a, b.$$

If we initially (at $t=0$) start at the point (x_0, y_0), then the parametric equation of the line is

$$x = x_0 + at, \quad y = y_0 + bt.$$

Slope of Parametric Curves

Suppose we have a parametric curve

$x = f(t), \quad y = g(t)$

and we also know that y can be expressed as a fn $h(x)$ of x.

Then by the chain rule (Leibniz form)

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

and if we divide by $\frac{dx}{dt}$ (provided $\frac{dx}{dt} \neq 0$).
we get.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

One can remember this formula by cancelling the dt's.

Similarly

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Ex. $x = \cos t, \quad y = \sin t, \qquad 0 \le t \le 2\pi$

unit circle..

Here $\frac{dx}{dt} = -\sin t \; , \; \frac{dy}{dt} = \cos t$

and so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -y/x$$

$$= -\cot t \qquad (t \neq 0, \pi, 2\pi).$$

For example at $t = \frac{\pi}{4}$, $\quad x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

and $\frac{dy}{dx} = -\cot\frac{\pi}{4} = -1.$

while $\frac{dx}{dt} = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}, \; \frac{dy}{dt} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

The parametric eqn of the tgt line is then given by

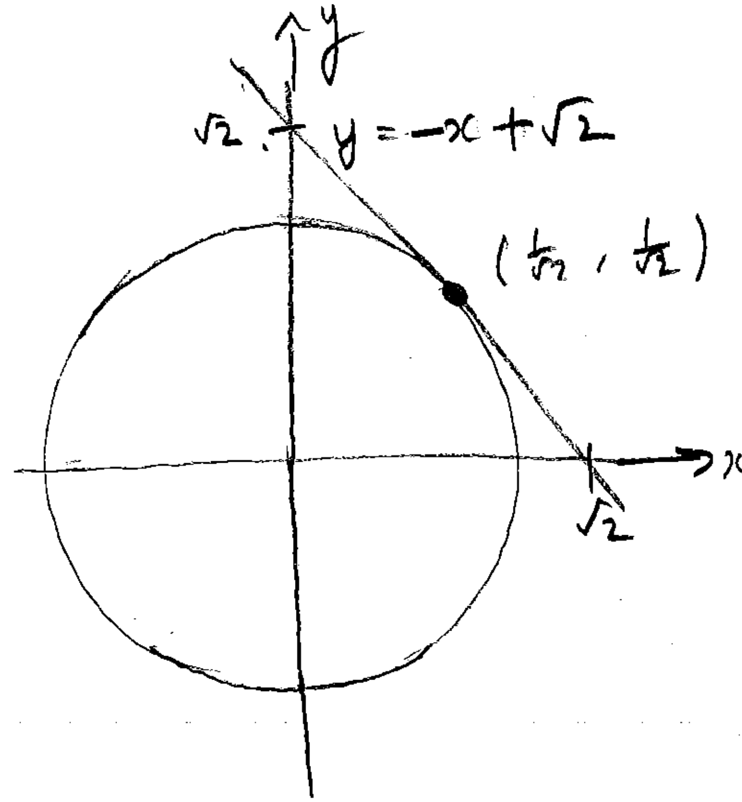
$$x = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} t, \quad y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} t$$

While the more usual form of the tgt line is given by

$$\left(y - \frac{1}{\sqrt{2}}\right) = (-1)\left(x - \frac{1}{\sqrt{2}}\right)$$

$$y = x + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$y = x + \sqrt{2}.$$



Parametric Curves

For a parametric curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, we have

$$\Delta x \approx \frac{dx}{dt} \Delta t = f'(t) \Delta t$$

$$\Delta y \approx \frac{dy}{dt} \Delta t = g'(t) \Delta t.$$

$$\begin{aligned}\Delta s &= \sqrt{(\Delta x)^2 + (\Delta y)^2} \\ &\approx \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, \Delta t \\ &= \sqrt{(f'(t))^2 + (g'(t))^2}\, \Delta t\end{aligned}$$

This leads to the integral

$$S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

If $x = f(t)$, $y = g(t)$, measures the displacement of a particle as a fn of time t, then the speed $v(t)$ is given by

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

and so the distance travelled, s, is given by

$$S = \int_a^b v(t)\,dt \qquad \text{i.e.} \qquad \text{distance} = \int_a^b \text{speed}\; dt.$$

Ex. For the circle $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, we have

$$\frac{dx}{dt} = -2\sin t, \quad \frac{dy}{dt} = 2\cos t$$

so

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{(-2\sin t)^2 + (2\cos t)^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{4(\sin^2 t + \cos^2 t)}$$

$$= \sqrt{4} \qquad \text{as } \sin^2 t + \cos^2 t = 1$$

$$= 2$$

and the circumference (distance travelled) is

$$S = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} 2\, dt = \left[2t\right]_0^{2\pi}$$

$$= 4\pi$$

as we'd expect.