§ 11.7 Models of Population Growth

We have already seen the exponential model of population growth

$$P = P_0 e^{kt}$$

P_0 is the initial population and k determines the growth rate of the population.

Usually the hard part is to find the value of k.

Ex. The pop. of Mexico in 1980 was about 67 million and in 1984 it was about 75 million. Find an exponential fn which models Mexico's pop. growth and use it to predict when the population is double that in 1980 (ie 134 million).

Let us measure P in millions of people and t in years since 1980.

$$P = P_0 e^{kt}$$

Here $P_0 = 67$ and so

$$P = 67 e^{kt}$$

In 1984, $t = 4$ and $P = 75$, so

$$75 = 67 e^{4k}$$

$$\frac{75}{67} = e^{4k}$$

Take ln of both sides

$$\ln(75/67) = 4k$$

$$k = \frac{\ln(75/67)}{4} \approx .0282.$$

Thus

$$P = 67e^{.0282t}$$

If the pop. is double the amount in 1980, we have $2(67) = 134$ million people.

Set

$$134 = 67e^{.0282t}$$

and solve for t.

$$\frac{134}{67} = e^{.0282t}$$

$$2 = e^{.0282t}$$

Take ln of both sides.

$$\ln 2 = \ln(e^{.0282t}) = .0282t$$

$$t = \frac{\ln 2}{.0282} \approx 24.57 \text{ years.}$$

Thus the pop. has doubled by around 2004/5.

As we saw in Mth 141, the trick to these problems is taking the ln of both sides and using the laws of logarithms.

Of course, the exponential model is rather unrealistic as it assumes the pop. is in an environment with unlimited resources for growth.

A more realistic model of pop. growth is the <u>logistic model</u>.

The starting point for this is the DE

$$\frac{dP}{dt} = kP(1 - P/L), \qquad L, k > 0$$

Note that if $P = L$, $\frac{dP}{dt} = 0$, so the constant fn $P = L$ is an equilibrium sol^n.

If $P < L$, $1 - P/L > 0 \Rightarrow \frac{dP}{dt} = kP(1 - P/L) > 0$

$P > L$, $1 - P/L < 0 \Rightarrow \frac{dP}{dt} = kP(1 - P/L) < 0.$

Thus if $P < L$, P will increase and if $P > L$, P will decrease and so we expet that this equilibrium should be stable.

In order do solve the DE, we first need the partial fractions expansion for

$$\frac{1}{P(1 - P/L)} = \frac{L}{P(L - P)}$$

(we'll see just why we need to do this in a few minutes)

So, let

$$\frac{L}{P(L-P)} = \frac{A}{P} + \frac{B}{L-P}$$

and solve for A, B.

Common denominator

$$\frac{L}{P(L-P)} = \frac{A(L-P)}{P(L-P)} + \frac{BP}{P(L-P)}$$

Multiply by $P(L-P)$

$$L = A(L-P) + BP$$

$$L = AL - AP + BP$$

$$= AL + (B-A)P$$

Equating the coefficients of these two polynomials (in P) gives

<u>Const term</u> $\quad L = AL \quad \Rightarrow \quad A = 1$

<u>P term</u> $\quad 0 = B - A \quad \Rightarrow \quad B = A = 1.$

Thus

$$\frac{1}{P(1 - P/L)} = \frac{1}{P} + \frac{1}{L - P}$$

Now back to our DE

$$\frac{dP}{dt} = kP(1 - P/L).$$

Separating the variables gives

$$\frac{dP}{P(1 - P/L)} = k\,dt$$

and if we now use our partial fractions expansion, we get

$$\left(\frac{1}{P} + \frac{1}{L-P}\right) dP = k\,dt$$

Now integrate both sides

$$\int \left(\frac{1}{P} + \frac{1}{L-P}\right) dP = \int k\,dt$$

$$\ln|P| - \ln|L-P| = kt + C, \quad C \text{ constant}$$

Take minus of both sides

$$\ln|L-P| - \ln|P| = -kt - C$$

Use laws of logarithms

$$\ln\left|\frac{L-P}{P}\right| = -kt - C$$

Take the exponential of both sides

$$e^{\ln\left|\frac{L-p}{p}\right|} = e^{-kt-c}$$

$$\left|\frac{L-p}{p}\right| = e^{-c}e^{-kt}$$

$$\frac{L-p}{p} = Ae^{-kt} \quad \text{where } A = \pm e^{-c}$$

(another constant).

$$\frac{L}{p} - 1 = Ae^{-kt}$$

$$\frac{L}{p} = 1 + Ae^{-kt}$$

$$\frac{p}{L} = \frac{1}{1+Ae^{-kt}}$$

$$p = \frac{L}{1+Ae^{-kt}}$$

To find A, use the IC, $P = P_0$ when $t = 0$.

$$P_0 = \frac{L}{1 + Ae^0}$$

$$= \frac{L}{1 + A}.$$

$$\frac{P_0}{L} = \frac{1}{1 + A}$$

$$\frac{L}{P_0} = 1 + A$$

$$A = \frac{L}{P_0} - 1.$$

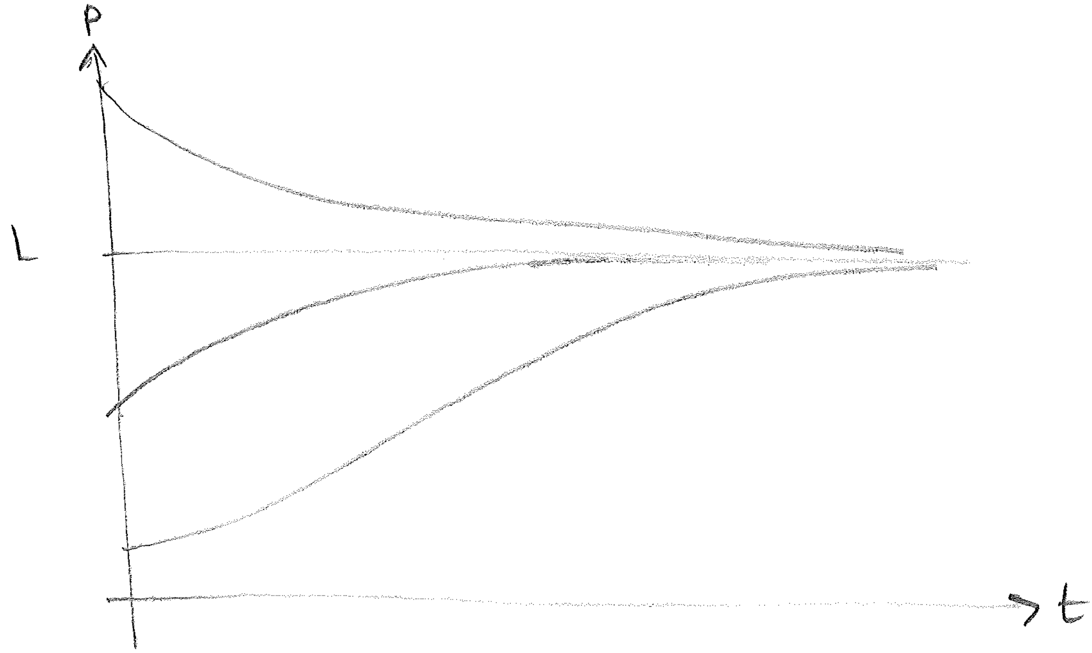
Thus the sol^{n} of the IVP,

$$\frac{dP}{dt} = kP(1 - P/L), \quad P(0) = P_0$$

is

$$P = \frac{L}{1 + Ae^{-kt}}, \quad \text{where} \quad A = \frac{L}{P_0} - 1.$$

Graphing solutions for different values of P_0 (but the same values of k, L) gives



The graph also suggests that the equilibrium L is stable.

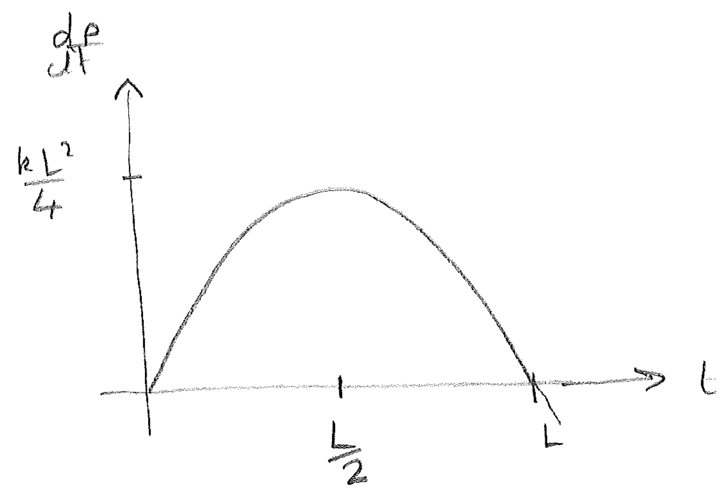
This can be seen more rigorously by noting that as $Ae^{-kt} \to 0$ as $t \to \infty$,

$$P(t) = \frac{L}{1 + Ae^{-kt}} \longrightarrow \frac{L}{1+0} = L \quad \text{as } t \to \infty.$$

If we go back to the original DE

$$\frac{dP}{dt} = kP(1 - P/L)$$

and plot $\frac{dP}{dt}$ as a fn of P, we get a quadratic (parabola).



Thus if $P < \frac{L}{2}$, $\frac{dP}{dt}$ is increasing and so P is concave up.

If $P > L/2$, $\frac{dP}{dt}$ is decreasing and so P is concave down.

Max. growth rate occurs at the point of inflection when $P = L/2$.