Math 142 Worksheet #2 Solutions

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1. i). $\int \sqrt{4-x^2}\, dx.$

Let $x = 2\sin\theta$, $dx = 2\cos\theta\, d\theta$, $\theta = \arcsin\left(\frac{x}{2}\right)$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$$
$$= \sqrt{4(1-\sin^2\theta)}$$
$$= 2\sqrt{1-\sin^2\theta}$$
$$= 2\sqrt{\cos^2\theta}$$
$$= 2\cos\theta.$$

<u>Rewrite</u>

$$\int \sqrt{4-x^2}\, dx = \int 2\cos\theta \cdot 2\cos\theta\, d\theta$$
$$= 4\int \cos^2\theta\, d\theta.$$

$$= 4 \int \frac{1}{2}(1 + \cos 2\theta)\,d\theta$$

using the half-angle identity

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$= 2 \int (1 + \cos 2\theta)\,d\theta$$

$$= 2\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

$$= 2\theta + \sin 2\theta + C$$

$$= 2 \arcsin \frac{x}{2} + \sin\left(2 \arcsin\left(\frac{x}{2}\right)\right) + C$$

Can also say

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$= 2 \sin\theta \sqrt{1 - \sin^2\theta}$$

$$= x\sqrt{1 - (x/2)^2} = x\sqrt{1 - x^2/4}$$

so we can rewrite the answer as

$$2 \arcsin \frac{x}{2} + x\sqrt{1 - x^2/4} + C.$$

ii) $\int \sqrt{9+4x^2}\, dx = \int \sqrt{3^2+(2x)^2}\, dx$

Use tangent subst.

$2x = 3\tan\theta$, $\theta = \arctan\left(\frac{2x}{3}\right)$.

$2dx = 3\sec^2\theta\, d\theta$

$dx = \frac{3}{2}\sec^2\theta\, d\theta$

$\sqrt{9+4x^2} = \sqrt{9+9\tan^2\theta}$

$= 3\sqrt{1+\tan^2\theta}$

$= 3\sqrt{\sec^2\theta}$ using $\sec^2\theta = 1+\tan^2\theta$.

$= 3\sec\theta$.

Rewrite.

$\int \sqrt{9+4x^2}\, dx = \int 3\sec\theta \cdot \frac{3}{2}\sec^2\theta\, d\theta$

$= \frac{9}{2}\int \sec^3\theta\, d\theta$.

No need to go any further than this!
(Although it can be done by parts + partial fractions).

iii)

$$\int \frac{1}{\sqrt{4x - x^2}}\, dx.$$

Complete the square.

$$4x - x^2 = -(x^2 - 4x)$$

$$= -(x^2 + 2(-2)x)$$

$$= -(x^2 + 2(-2)x + (-2)^2 - (-2)^2)$$

$$= -((x-2)^2 - 4)$$

$$= 4 - (x-2)^2$$

So $\displaystyle\int \frac{1}{\sqrt{4x - x^2}}\, dx = \int \frac{1}{\sqrt{4-(x-2)^2}}\, dx.$

Now let $x - 2 = 2\sin\theta.$

Then $\theta = \arcsin\left(\frac{x}{2} - 1\right)$

$dx = 2\cos\theta \, d\theta$

and $\sqrt{4-(x-2)^2} = \sqrt{4 - 4\sin^2\theta} = 2\cos\theta.$

Rewrite the integral as.

$$\int \frac{1}{2\cos\theta} \cdot 2\cos\theta \, d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arcsin\left(\frac{x}{2} - 1\right) + C.$$

iv) $\displaystyle\int \frac{1}{(10+4x+4x^2)^{3/2}}$

Complete the square.

$$4x^2+4x+10 = 4\left(x^2+x+\frac{10}{4}\right)$$

$$= 4\left(x^2+2\left(\tfrac{1}{2}\right)x+\left(\tfrac{1}{2}\right)^2-\left(\tfrac{1}{2}\right)^2+\tfrac{10}{4}\right)$$

$$= 4\left(\left(x+\tfrac{1}{2}\right)^2+\tfrac{9}{4}\right)$$

$$= (2x+1)^2+3^2.$$

Rewrite

$$\int \frac{1}{(10+4x+4x^2)^{3/2}}\,dx = \int \frac{1}{((2x+1)^2+3^2)^{3/2}}\,dx$$

Suggests tan subst.

$$2x+1 = 3\tan\theta$$

$$\theta = \arctan\left(\frac{2x+1}{3}\right).$$

$$2\,dx = 3\sec^2\theta\,d\theta$$

$$dx = \frac{3}{2}\sec^2\theta\,d\theta$$

$$\left((2x+1)^2+3^2\right)^{\frac{3}{2}}$$

$$= \left(9\tan^2\theta + 9\right)^{\frac{3}{2}}$$

$$= 9^{\frac{3}{2}}\left(1+\tan^2\theta\right)^{\frac{3}{2}}$$

$$= 9^{\frac{3}{2}}\left(\sec^2\theta\right)^{\frac{3}{2}} \qquad \text{as} \quad \sec^2\theta = 1+\tan^2\theta$$

$$= 27\sec^3\theta.$$

Rewrite.

$$\int \frac{1}{\left((2x+1)^2+3^2\right)^{\frac{3}{2}}}\,dx$$

$$= \int \frac{1}{27\sec^3\theta}\cdot\frac{3}{2}\sec^2\theta\,d\theta$$

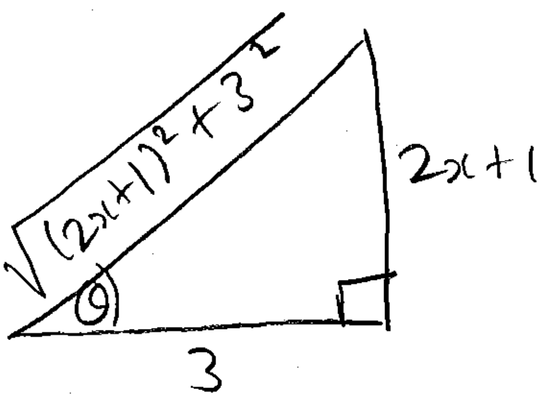
$$= \frac{1}{18} \int \frac{d\theta}{\sec\theta}$$

$$= \frac{1}{18} \int \cos\theta \, d\theta$$

$$= \frac{1}{18} \sin\theta + C$$

Use a right-angled triangle to figure out $\sin\theta$ in terms of x using $\tan\theta = \frac{2x+1}{3}$.

By Pythageras and defn of sine



$$\sin\theta = \frac{opp}{hyp}$$

$$= \frac{2x+1}{\sqrt{(2x+1)^2+3^2}}$$

$$= \frac{2x+1}{\sqrt{10+4x+4x^2}}$$

Then finally (!)

$$\int \frac{dx}{(10+4x+4x^2)^{\frac{3}{2}}}$$

$$= \frac{1}{18} \sin\theta + C$$

$$= \frac{1}{18} \cdot \frac{2x+1}{\sqrt{10+4x+4x^2}} + C$$

$$= \frac{2x+1}{18\sqrt{10+4x+4x^2}} + C$$

2. P.F.E = Partial Fractions Expansion

i) $\dfrac{1}{(x-3)(x+7)}$ P.F.E: $\dfrac{A}{x-3} + \dfrac{B}{x+7}$

ii) $\dfrac{2x+5}{(x-3)^2(x-7)^3}$

P.F.E. $\dfrac{A}{(x-3)} + \dfrac{B}{(x-3)^2} + \dfrac{C}{x-7} + \dfrac{D}{(x-7)^2} + \dfrac{E}{(x-7)^3}$

iii) $\dfrac{x^2+7x+2}{(x^2+4)(x-1)^4}$

P.F.E.

$$\frac{Ax+B}{x^2+4} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3} + \frac{F}{(x-1)^4}.$$

3. i) $\int \frac{dx}{x^2-1}$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}.$$

So $$\frac{1}{(x-1)(x+1)} = \frac{(A+B)x + A-B}{(x-1)(x+1)}$$

Comparing like powers of x.

x/ $\quad A+B = 0 \quad$ ①

1/ $\quad A-B = 1 \quad$ ②

① $\Rightarrow \quad B = -A$

Sub in ② $\quad A-(-A) = 1$

$2A = 1$

$A = \frac{1}{2}$ so $B = -A = -\frac{1}{2}$.

$A = \frac{1}{2}$, $B = -\frac{1}{2}$.

Thus

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

and

$$\int \frac{dx}{x^2-1} = \int \left\{ \frac{1}{2(x-1)} - \frac{1}{2(x+1)} \right\} dx$$

$$= \frac{1}{2} \int \frac{dx}{x-1} - \frac{1}{2} \int \frac{dx}{x+1}$$

$$= \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| + C.$$

ii) $\displaystyle\int \frac{x^2+2x-1}{(x-1)(x+1)^2}\,dx.$

Write.

$$\frac{x^2+2x-1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

$$= \frac{A(x+1)^2 + B(x-1)(x+1) + C(x-1)}{(x-1)(x+1)^2}$$

$$= \frac{A(x^2+2x+1) + B(x^2-1) + C(x-1)}{(x-1)(x+1)^2}.$$

$$\frac{x^2+2x-1}{(x-1)(x+1)^2}. = \frac{(A+B)x^2 + (2A+C)x + A-B-C}{(x-1)(x+1)^2}.$$

Compare like powers of x.

$$\begin{aligned} A + B \quad &= 1 \quad (1) \\ 2A \quad + C &= 2 \quad (2) \\ A - B - C &= -1 \quad (3) \end{aligned}$$

Gaussian Elimination

Subtract 2 (1) from 2 and 1 (1) from (3). to eliminate A from (2), (3)

$$\begin{aligned} A + B \quad &= 1 \quad (1) \\ -2B + C &= 0 \quad (2)' \\ -2B - C &= -2 \quad (3)' \end{aligned}$$

Subtract (2) from (3) to eliminate B in (3)'

$$\begin{aligned} A + B \quad &= 1 \quad (1) \\ -2B + C &= 0 \quad (2)' \\ -2C &= -2 \quad (3)' \end{aligned}$$

(3)'' $\Rightarrow$ $C = 1$

Sub in (2)'

$$-2B + 1 = 0$$

$$-2B = -1$$

$$B = \frac{1}{2}.$$

Sub for B, C in (1).

$$A + \frac{1}{2} = 1$$

$$A = \frac{1}{2}.$$

So $A = \frac{1}{2}$, $B = \frac{1}{2}$, $C = 1$.

Thus

$$\frac{x^2 + 2x - 1}{(x-1)(x+1)^2} = \frac{1}{2(x-1)} + \frac{1}{2(x+1)} + \frac{1}{(x+1)^2}$$

Hence.

$$\int \frac{x^2+2x-1}{(x-1)(x+1)^2}\,dx = \int \left\{ \frac{1}{2(x-1)} + \frac{1}{2(x+1)} + \frac{1}{(x+1)^2} \right\} dx$$

$$= \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2}$$

$$= \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{x+1} + C$$

iii) $$\int \frac{2x^2}{(x-1)(x^2+1)}$$

x^2+1 is irreducible and the P.F.E is

$$\frac{2x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{A(x^2+1) + Bx(x-1) + C(x-1)}{(x-1)(x^2+1)}$$

$$= \frac{A(x^2+1) + B(x^2-x) + C(x-1)}{(x-1)(x^2+1)}$$

$$\frac{2x^2}{(x-1)(x^2+1)} = \frac{(A+B)x^2 + (-B+C)x + A-C}{(x-1)(x^2+1)}.$$

Compare like powers of x.

$$
\begin{array}{llcl}
\underline{x^2} & A + B & = 2 & ① \\
\underline{x} & \quad - B + C & = 0 & ② \\
\underline{1} & A \qquad - C & = 0 & ③
\end{array}
$$

Gaussian Elimination

Take ① away from ③ to get rid of A in ③.

$$
\begin{array}{lcl}
A + B & = 2 & ① \\
\quad - B + C & = 0 & ② \\
\quad - B - C & = +2 & ③'
\end{array}
$$

Subtract ② from ③' to get rid of B in ③'

$$
\begin{array}{lcl}
A + B & = 2 & ① \\
\quad - B + C & = 0 & ② \\
\qquad - 2C & = -2 & ③
\end{array}
$$

③ $\Rightarrow$ $C = 1$

Sub in 2

$$-1 + 1 = 0 \Rightarrow B = 1$$

Sub for B, C in ①

$$A + 1 = 2 \Rightarrow A = 1.$$

So $A = B = C = 1$ and

$$\frac{2x^2}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{x+1}{x^2+1}$$

Then

$$\int \frac{2x^2}{(x-1)(x^2+1)}\,dx = \int \left\{ \frac{1}{x-1} + \frac{x+1}{x^2+1} \right\} dx$$

$$= \int \frac{dx}{x-1} + \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$$

For second integral, let $w = x^2+1$

so $dw = 2x\,dx$

$$\frac{dw}{2} = dx.$$

$$= \int \frac{dx}{x-1} + \int \frac{\frac{dw}{2}}{w} + \int \frac{dx}{x^2+1}$$

$$= \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dw}{w} + \int \frac{dx}{x^2+1}$$

$$= \ln|x-1| + \frac{1}{2} \ln|w| + \arctan x + C$$

$$= \ln|x-1| + \frac{1}{2} \ln|x^2+1| + \arctan x + C.$$