III. (12 pts.) Use either the Integral Test OR the Comparison Test to determine whether the series below are convergent or divergent. Explain.

a). $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ 6 each

b). $\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$ ok ⇒ div, b 2

a). $\int$ test $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx$ $w = \ln x,\ dw = \frac{dx}{x}$, $x = 0,\ w = 1$

$= \lim_{b\to\infty} \int_0^{\ln b} w\,dw = \lim_{b\to\infty} \left[\frac{w^2}{2}\right]_0^{\ln b} = \lim_{b\to\infty} (\ln b)^2 - 0 = \infty$

div by $\int$ test only 2.

b). Comp. test. $0 \le 1+\sin n \le 2$, so

$0 \le \frac{1+\sin n}{10^n} \le \frac{2}{10^n}$ and as $\sum_{n=0}^{\infty} \frac{2}{10^n}$ is a conv. geom series, $\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$ conv by comp. test only 2

IV. (12 pts.) Use either the Alternating Series Test OR the Ratio Test to determine whether the series below are convergent or divergent. Explain.

a). $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+1)^2}$

b). $\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$ 6 each

a) Alt. series test. Here $a_n = \frac{1}{(n+1)^2}$

$a_n \to 0$ as $n \to \infty$ and $0 < a_{n+1} < a_n$.

Thus the series is conv.

b) Ratio test. $a_n = \frac{3^n}{(2n)!}$, $a_{n+1} = \frac{3^{n+1}}{(2(n+1))!} = \frac{3^{n+1}}{(2n+2)!} = \frac{3^{n+1}}{(2n+2)(2n+1)\cdot(2n)!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(2n+2)(2n+1)(2n)!} \Big/ \frac{3^n}{(2n)!} = \frac{3}{(2n+2)(2n+1)} \to 0$ as $n \to \infty$

Series converges by ratio test $(0 < 1)$.

V. (12 pts.) Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

$a_n = \dfrac{(-3)^n x^n}{\sqrt{n+1}}$, $a_{n+1} = \dfrac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}}$.

$\left|\dfrac{a_{n+1}}{a_n}\right| = \left| \dfrac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \Big/ \dfrac{(-3)^n x^n}{\sqrt{n+1}} \right|$ no denom by n -1

$= |x| \cdot 3 \dfrac{\sqrt{n+1}}{\sqrt{n+2}} = |x| \cdot 3 \dfrac{\sqrt{1+\frac{1}{n}}}{\sqrt{1+2/n}} \to 3|x|$ as $n \to \infty$.

Thus $R = \frac{1}{3}$. 6

At $x = \frac{1}{3}$, have

3

$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ $\frac{1}{\sqrt{n+1}} \to 0$ as $n \to \infty$ and is decr in n.

Conv. by alt. series test

At $x = -\frac{1}{3}$, have

$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$, comp with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

3

$\frac{1}{\sqrt{n+1}} \Big/ \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1+\frac{1}{n}}} \to 1$ as $n \to \infty$.

Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ and hence $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ diverge by limit comp. test

Int. of convergence $(-\frac{1}{2}, \frac{1}{2}]$.

Taylor coffs 4 rest 3.

VI. (14 pts.) New series from old. Find the first 5 non-zero terms of a power series representation about $a = 0$ for the functions:

a). $f(x) = \dfrac{x^3}{1+2x}$ 7 each. Taylor coeffs. ok! ~~max 5/7~~

b). $f(x) = x\sin(x^2)$

missing -2

$$\frac{x^3}{1+2x} = \frac{x^3}{1-(-2x)} \approx x^3\left(1-2x+4x^2-8x^3+16x^4\right)$$

$$= x^3 - 2x^4 + 4x^5 - 8x^6 + 16x^7$$

$$x\sin(x^2) \approx x\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!}\right)$$

$$= x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \frac{x^{19}}{9!}$$

VII. (12 pts.) Use the Taylor series expansion to find the first 4 terms of the Taylor series for the function

New from old - ok!

$$f(x) = \frac{1}{x} \qquad \text{about} \qquad a = 2$$

$f(x) = \frac{1}{x}$, $f(2) = \frac{1}{2}$

$f'(x) = -\frac{1}{x^2}$, $f'(2) = -\frac{1}{4}$

$f''(x) = \frac{2}{x^3}$, $f''(2) = \frac{2}{8} = \frac{1}{4}$

$f'''(x) = -\frac{6}{x^4}$, $f'''(2) = -\frac{6}{16}$
$= -\frac{3}{8}$.

8

$$\frac{1}{x} \approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{\frac{1}{4}}{2!}(x-2)^2 - \frac{3/8}{3!}(x-2)^3 \quad \cancel{6}\ 4$$

here only -1

$$= \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

VIII. (14 pts.)

a). Use the Taylor series expansion to find the Taylor polynomial of degree 3 about $a = 0$ for

$$f(t) = 2e^{3t}$$

b). Give a bound on the interval $-1/2 \le x \le 1/2$ for the error E_3. Explain.

7/ a). $f(t) = 2e^{3t}$, $f(0) = 2$, $f'''(t) = 54e^{3t}$, $f'''(0) = 54$

$f'(t) = 6e^{3t}$, $f'(0) = 6$, $4 = 4 \times 1$

$f''(t) = 18e^{3t}$, $f''(0) = 18$

$$2e^{3t} \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= 2 + 6x + \frac{18}{2!}x^2 + \frac{54}{6}x^3$$ 3

$$= 2 + 6x + 9x^2 + 9x^3.$$

b) $f^{(4)}(t) = 162e^{3t}$ 2

Has max value on $[-\frac{1}{2}, \frac{1}{2}]$ at $\frac{1}{2}$ as it is incr. - max value is $162e^{3/2}$. 2

By Lagrange, if we let $M = 162e^{3/2}$, M for f instead of $f^{(n+1)}$ -2

3

$$|E_3| \le \frac{Mx^4}{4!} = \frac{162e^{3/2}}{24}x^4 = \frac{27}{4}e^{3/2}x^4$$