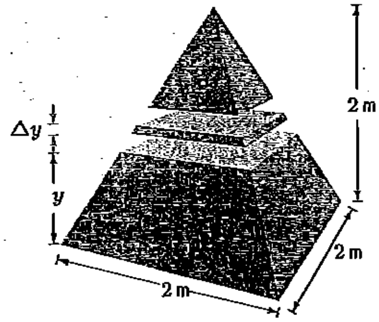
III. (13 pts.) Write a definite integral representing the volume of the region, using the slice shown. Evaluate the integral exactly.



$A(y) = (s(y))^2$

$= (2-y)^2$

$s = a + by$

$s(0) = 2 \Rightarrow a = 2$

$s(2) = 0 \Rightarrow 2 + 2b = 0$

$b = -1$

$s(y) = 2 - y.$

$V = \int_0^2 A(y)\,dy$

$= \int_0^2 (2-y)^2\,dy$

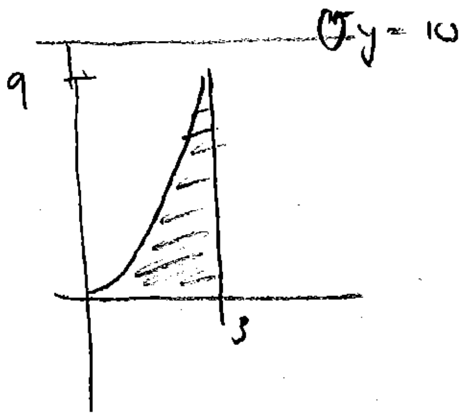
$= \int_0^2 (4 - 4y + y^2)\,dy$

$= \left[4y - 2y^2 + y^3/3\right]_0^2$

$= 8 - 8 + \frac{8}{3} - 0$

$= \frac{8}{3}\ m^3$

IV. (10 pts.) Set up and compute an integral giving the volume of the solid of revolution bounded by $y = x^2$, the x-axis, $x = 0$, $x = 3$, rotated about $y = 10$.



$V = \pi \int_0^3 \left(10^2 - (10 - x^2)^2\right)dx$

$= \pi \int_0^3 \left(100 - (100 - 20x^2 + x^4)\right)dx$

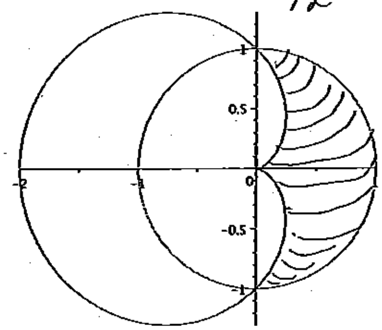
$= \pi \int_0^3 (20x^2 - x^4)\,dx$

$= \pi \left[\frac{20x^3}{3} - \frac{x^5}{5}\right]_0^3 = \pi\left(\frac{20 \times 27}{3} - \frac{243}{5} - 0\right)$

$= \pi\left(180 - \frac{245}{3}\right) = \frac{195\pi}{3}$

V. (13 pts.) Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

$\pi/2$



$-\pi/2$

$$A = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1^2 - (1-\cos\theta)^2\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - (1 - 2\cos\theta + \cos^2\theta)\right) d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(2\cos\theta - \frac{1}{2} - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\left(2 - \frac{\pi}{4} - 0\right) - \left(-2 + \frac{\pi}{4} - 0\right)\right)$$

$$= \frac{1}{2}\left(4 - \frac{\pi}{2}\right) = 2 - \frac{\pi}{4}$$

VI. (10 pts.) A rod of length 2 meters and density $\delta(x) = 3 - e^{-x}$ kilograms per meter is placed on the x-axis with its left end at the origin. Find the center of mass of the rod.

$$M = \int_0^2 \delta(x)\,dx = \int_0^2 (3 - e^{-x})\,dx = \left[3x + e^{-x}\right]_0^2 = (6 + e^{-2}) - (e^0)$$
$$= 5 + e^{-2} \text{ kg.}$$

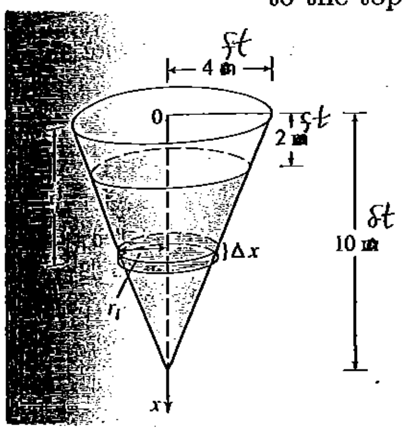
$$M_x = \int_0^2 x\delta(x)\,dx = \int_0^2 (3x - xe^{-x})\,dx = \int_0^2 3x\,dx - \int_0^2 xe^{-x}\,dx$$

$$= \int_0^2 3x\,dx - \left\{\left[-xe^{-x}\right]_0^2 + \int_0^2 e^{-x}\,dx\right\} = \left[3\frac{x^2}{2}\right]_0^2 + \left[xe^{-x}\right]_0^2 + \left[-e^{-x}\right]_0^2$$

$$= 6 + 2e^{-2} + e^{-2} - 1 = 5 + 3e^{-2} \text{ kg m}$$

$$\bar{x} = \frac{M_x}{M} = \frac{5 + 3e^{-2}}{5 + e^{-2}}$$

VII. (14 pts.) A tank has the shape of an inverted circular cone with height 10 ft and base radius 4 ft. It is filled with water to a height of 8 ft. Find the work required to empty the tank by pumping all of the water to the top of the tank. (The density of the water is $\delta = 62.4 \text{lb/ft}^3$.)



$r(h) = ah + b$ $\quad r(0) = 4 \Rightarrow b = 4$

$r(10) = 0 \Rightarrow 4 + 10a = 0 \quad a = -\frac{2}{5}$

$r(h) = 4 - \frac{2h}{5}$

$A(h) = \pi r^2 = \pi\left(4 - \frac{2h}{5}\right)^2 = \pi\left(16 - \frac{16}{5}h + \frac{4h^2}{25}\right)$

$\Delta V = A(h)\Delta h = \pi\left(16 - \frac{16h}{5} + \frac{4h^2}{25}\right)\Delta h$

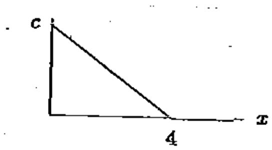
$\Delta F = 62.4\,\Delta V = 62.4\,\pi\left(16 - \frac{16h}{5} + \frac{4h^2}{25}\right)$

$\Delta W = h\Delta F = 100(9.8)\pi\left(16h - \frac{16h^2}{5} + 4h^3/25\right)$

$W = 62.4\,\pi \int_2^{10}\left(16h - \frac{16}{5}h^2 + 4h^3/25\right)dh = 62.4\,\pi\left[8h^2 - \frac{16h^3}{15} + \frac{h^4}{25}\right]_2^{10}$

$= 62.4\pi\left(\left(800 - \frac{16000}{15} + \frac{10000}{25}\right) - \left(32 - \frac{128}{15} + \frac{16}{25}\right)\right) = 62.4\pi\left(5000 - \frac{8000}{3} + 400 - \left(200 - \frac{64}{3} + \frac{16}{25}\right)\right)$

VIII. (10 pts.) Decide if the function graphed is a probability density function (pdf) or a cumulative distribution function (cdf). Give reasons. Find the value of c. Sketch and label the cdf if the problem shows a pdf, and the pdf if the problem shows a cdf.



Fn is decreasing somewhere

$\Rightarrow$ cannot be a CDF.

So must be a pdf.

$p(x) = a + bx$ on $[0, 4]$ $\quad p(0) = c \quad p(4) = 0$

$p(0) = c \Rightarrow a = c \quad p(b) = 4 \Rightarrow c + 4b = 0$

$b = -\frac{c}{4}$

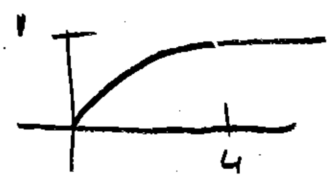
$p(x) = c - \frac{c}{4}x = c(1 - x/4)$

$\int_{-\infty}^{\infty} p(x)\,dx = 1 \Rightarrow \int_0^4 c(1 - x/4)\,dx = 1$

$\Rightarrow c\left[x - x^2/8\right]_0^4 = 1 \Rightarrow c(4 - 2 - 0) = 1$

$2c = 1$

$c = \frac{1}{2}$



CDF.

$$P(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}(t - t^2/8), & 0 \le t \le 4 \\ 1, & t > 4. \end{cases}$$

IX. (10 pts.) Let $p(t) = 0.1e^{-0.1t}$ be the density function for the waiting time at a subway stop, with t in minutes, $0 \le t \le 60$.

a). What proportion of people wait at a subway stop between $0 \le t \le 20$ minutes?

b). What is the mean time for a person to wait at a subway stop?

c). Compute the median of this distribution.

a). $$\int_0^{20} .1e^{-.1t}\,dt = \left[-e^{-.1t}\right]_0^{20} = 1 - e^{-2}$$

b) $$\int_0^{\infty} .1te^{-.1t}\,dt = \lim_{b\to\infty} \int_0^b .1te^{-.1t}\,dt$$

$$= \lim_{b\to\infty} \left\{ \left[-te^{-.1t}\right]_0^b + \int_0^b e^{-.1t}\,dt \right\}$$

$$= \lim_{b\to\infty} \left\{ \left[-te^{-.1t}\right]_0^b + \left[-10e^{-.1t}\right]_0^b \right\}$$

$$= \lim_{b\to\infty} \left\{ -be^{-.1b} - 0 + -10e^{-.1b} - (-10) \right\}$$

$= 10$ minutes.

c). $$\int_0^T .1e^{-.1t}\,dt = \frac{1}{2} \qquad \left[-e^{-.1t}\right]_0^T = \frac{1}{2}$$

$$1 - e^{-.1T} = \frac{1}{2} \qquad e^{-.1T} = \frac{1}{2} \qquad -.1T = \ln\frac{1}{2}$$

$$T = \frac{\ln\frac{1}{2}}{-.1} = 10\ln 2.$$

$$\approx 6.93 \text{ min}$$

MTH - 142 [redacted]

Name:______________________ Student ID:______

Section:______ Instructor:______________________

SHOW ALL YOUR WORK! NO WORK-NO CREDIT!, NO CALCULATOR!, NO NOTES!

I. (12 pts.) Investigate if the sequences below converge or diverge. Find the limit of each convergent sequence. If the sequence does not converge explain why is that.

a). $a_n = (\frac{n^3+1}{2n^3})(1+\frac{100}{n})$

b). $a_n = \cos(\frac{3\pi}{2} - \frac{7}{n^3})$

c). $a_n = 1 + (-1)^{n+1}$

a) $a_n = \left(\frac{1+\frac{1}{n^3}}{2}\right)\left(1+\frac{100}{n}\right) \to \left(\frac{1+0}{2}\right)(1+0) = \frac{1}{2}$

as $n \to \infty$

b) $a_n = \cos\left(\frac{3\pi}{2} - \frac{7}{n^3}\right) \to \cos\left(\frac{3\pi}{2}\right) = 0$

as $n \to \infty$.

c) $a_n = 1+(-1)^{n+1} = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$ Diverges

II. (12 pts.) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

a). $\sum_{n=2}^{\infty}(\frac{7\cdot 3^n}{5^n}+\frac{4}{8^n})$

b). $\cancel{\sum_{n=4}^{\infty}\frac{n^2(n+5)}{(n+3)^3}}$

c). $b^9+b^{10}+b^{11}+b^{12}+b^{13}+b^{14}$

a) $= 7\sum_{n=2}^{\infty}\left(\frac{3}{5}\right)^n + 4\sum_{n=2}^{\infty}\frac{1}{8^n}$.

Both geometric series have ratio between -1 & 1 and so converge. The sum is then

$$7\left(\frac{3}{5}\right)^2\cdot\frac{1}{1-\left(\frac{3}{5}\right)} + 4\left(\frac{1}{8}\right)^2\cdot\frac{1}{1-\frac{1}{8}}$$

$$7\times\frac{9}{25}\times\frac{5}{2} + \frac{4}{64}\cdot\frac{8}{7} = \frac{63}{10}+\frac{1}{14}$$

c) $b^9+b^{10}+\cdots+b^{14}$ – finite geometric series (converges trivially)

$= b^9(1+b+\cdots+b^5)$

$= b^9\dfrac{(1-b^6)}{1-b}$