1. (17, p. 663) If $f\left(r,h\right)=2p\,rh+p\,r^2$, $\nabla f=\left[f_r \quad f_h\right]=\left[2p\,h+2p\,r \quad 2p\,r\right]$. At the point (2, 3), this is $\left[10p \quad 4p\right]=10p\,i+4p\,j$.
2. (27, p. 664) We know that the gradient is perpendicular to the contour and points in the direction of increase. From the point (2, 0), this would then point directly to the right in the *i*-direction.
3. (37, p. 664) If $f\left(x,y\right)=\sin\left(2x-y\right)$, $\nabla f=\left[f_x \quad f_y\right]=\left[2\cos\left(2x-y\right) \quad -\cos\left(2x-y\right)\right]$. At the point (1, 2), this is $\left[2 \quad -1\right]=2i-j$. Therefore,

   $$f_{\vec{u}}\left(1,2\right)=\left(2i-j\right)\cdot\vec{u}=\left(2i-j\right)\cdot\frac{\left(3i-4j\right)}{5}=\frac{1}{5}\left(2\cdot 3+\left(-1\right)\cdot\left(-4\right)\right)=2$$
4. (51, p. 665) If $f\left(x,y\right)=3xy+y^2$:
   1. $\nabla f=\left[3y \quad 3x+2y\right]$. At the point (2, 3), this is $\left[9 \quad 12\right]=9i+12j$. The unit vector in the direction of $3i-j$ is $\vec{u}=\frac{3i-j}{\sqrt{3^2+\left(-1\right)^2}}=\frac{3i-j}{\sqrt{10}}$. Therefore, the rate of change of *f* in this direction at this point is

      $$f_{\vec{u}}\left(2,3\right)=\left(9i+12j\right)\cdot\vec{u}=\left(9i+12j\right)\cdot\frac{\left(3i-j\right)}{\sqrt{10}}=\frac{1}{\sqrt{10}}\left(9\cdot 3+12\cdot\left(-1\right)\right)=\frac{15}{\sqrt{10}}.$$
   2. The direction of the maximum rate of change is just the direction of the gradient, $\frac{9i+12j}{\sqrt{9^2+12^2}}=\frac{9i+12j}{3\sqrt{3^2+4^2}}=\frac{3i+4j}{5}$.
5. (17, p. 671) Notice that since $\left(-1\right)^2-1^2+2^2=4$, (-1, 1, 2) lies on the surface. This is a level surface $w=4$ for the function $w=f\left(x,y,z\right)=x^2-y^2+z^2$. We know that the gradient of any function is perpendicular to its level sets, so that $\nabla f=\left[f_x \quad f_y \quad f_z\right]=\left[2x \quad -2y \quad 2z\right]$ is a normal vector, at every point. In particular, at (-1, 1, 2), we have the normal vector $\left[2\cdot\left(-1\right) \quad -2\cdot 1 \quad 2\cdot 2\right]=-2i-2j+4k$. The equation for the tangent plane is then $-2x-2y+4z=c$. Plugging in the point (-1, 1, 2), gives $c=-2\cdot\left(-1\right)-2\cdot 1+4\cdot 2=8$, so that the equation of the plane is $-2x-2y+4z=8$.
6. (24, p. 671) If $G\left(x,y,z\right)=x^2-5xy+y^2z$:
   1. $\nabla G=\left[2x-5y \quad -5x+2yz \quad y^2\right]$. At the point (1, 2, 3), this is $\left[2\cdot 1-5\cdot 2 \quad 5\cdot 1+2\cdot 2\cdot 3 \quad 2^2\right]=-8i+17j+4k$. The unit vector in the direction of $2i+j-4k$ is $\vec{u}=\frac{2i+j-4k}{\sqrt{2^2+\left(-1\right)^2+4^2}}=\frac{2i+j-4k}{\sqrt{21}}$. Therefore, the

rate of change of G in this direction at this point is

$$G_{\vec{u}}\left(2,3\right)=\left(-8i+17j+4k\right)\cdot\frac{2i+j-4k}{\sqrt{21}}$$
$$=\frac{1}{\sqrt{21}}\left(\left(-8\right)\cdot 2+17\cdot 1+4\cdot\left(-4\right)\right)=-\frac{15}{\sqrt{21}}.$$

b. The direction of the maximum rate of change is just the direction of the gradient, $\dfrac{-8i+17j+4k}{\sqrt{\left(-8\right)^2+17^2+4^2}}=\dfrac{-8i+17j+4k}{\sqrt{369}}$.

c. The maximum rate of change is the directional derivative in the direction of the gradient, $\left(-8i+17j+4k\right)\cdot\dfrac{-8i+17j+4k}{\sqrt{369}}=\dfrac{369}{\sqrt{369}}=\sqrt{369}$. Notice this turns out to be just the magnitude of the gradient, as it always will.

7. (5, p. 679) If $z=f\left(x,y\right)=xe^{y}$ and $\left(x,y\right)=g\left(t\right)=\left(2t,1-t^2\right)$, then $z=f\circ g\left(t\right)$ and the Chain Rule says that:

$$\frac{dz}{dt}=D\left(f\circ g\right)=Df\cdot Dg=\begin{bmatrix}\dfrac{dz}{dx} & \dfrac{dz}{dy}\end{bmatrix}\begin{bmatrix}\dfrac{dx}{dt}\\ \dfrac{dy}{dt}\end{bmatrix}=\begin{bmatrix}e^{y} & xe^{y}\end{bmatrix}\begin{bmatrix}2\\ -2t\end{bmatrix}$$
$$=e^{y}\cdot 2+xe^{y}\cdot\left(-2t\right)=2e^{y}\left(1-xt\right)=2e^{1-t^2}\left(1-\left(2t\right)t\right)=2e^{1-t^2}\left(1-2t^2\right)$$

8. (13, p. 679) If $z=f\left(x,y\right)=\sin\left(\dfrac{x}{y}\right)$ and $\left(x,y\right)=g\left(u,v\right)=\left(\ln\left(u\right),v\right)$, then $z=f\circ g\left(t\right)$ and the Chain Rule says that:

$$\begin{bmatrix}\dfrac{dz}{du} & \dfrac{dz}{dv}\end{bmatrix}=D\left(f\circ g\right)=Df\cdot Dg=\begin{bmatrix}\dfrac{dz}{dx} & \dfrac{dz}{dy}\end{bmatrix}\begin{bmatrix}\dfrac{dx}{du} & \dfrac{dx}{dv}\\ \dfrac{dy}{du} & \dfrac{dy}{dv}\end{bmatrix}$$
$$=\begin{bmatrix}\dfrac{1}{y}\cos\left(\dfrac{x}{y}\right) & -\dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)\end{bmatrix}\begin{bmatrix}\dfrac{1}{u} & 0\\ 0 & 1\end{bmatrix}$$
$$=\begin{bmatrix}\left(\dfrac{1}{y}\cos\left(\dfrac{x}{y}\right)\right)\cdot\left(\dfrac{1}{u}\right)+\left(-\dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)\right)\cdot 0 & \left(\dfrac{1}{y}\cos\left(\dfrac{x}{y}\right)\right)\cdot 0+\left(-\dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)\right)\cdot 1\end{bmatrix}$$
$$=\begin{bmatrix}\dfrac{1}{yu}\cos\left(\dfrac{x}{y}\right) & -\dfrac{x}{y^2}\cos\left(\dfrac{x}{y}\right)\end{bmatrix}=\begin{bmatrix}\dfrac{1}{vu}\cos\left(\dfrac{\ln\left(u\right)}{v}\right) & -\dfrac{\ln\left(u\right)}{v^2}\cos\left(\dfrac{\ln\left(u\right)}{v}\right)\end{bmatrix}$$

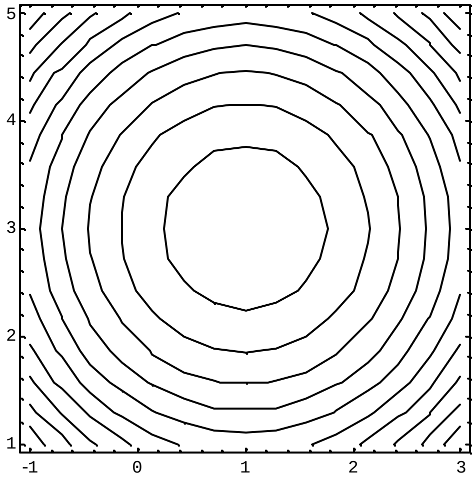
9. (5, p. 686) If $f\left(x,y\right)=\left(x+y\right)e^{y}$,

$Df=\begin{bmatrix}f_x & f_y\end{bmatrix}=\begin{bmatrix}e^{y} & e^{y}+\left(x+y\right)e^{y}\end{bmatrix}=\begin{bmatrix}e^{y} & \left(1+x+y\right)e^{y}\end{bmatrix}$, and so:

$$D^2 f = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 0 & e^y \\ e^y & e^y + (1+x+y)e^y \end{bmatrix} = \begin{bmatrix} 0 & e^y \\ e^y & (2+x+y)e^y \end{bmatrix}$$

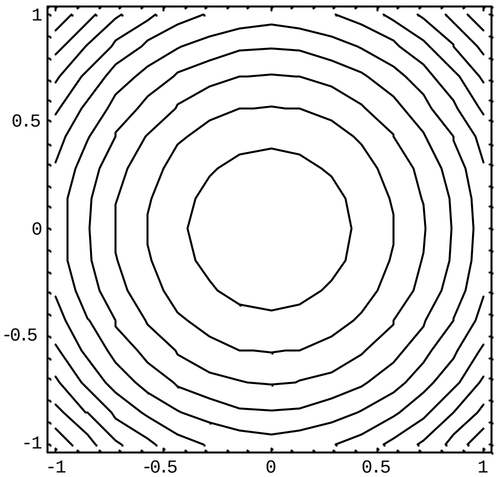
Notice that $f_{xy} = e^y = f_{yx}$.

10. (1, p.709) The point A is not a critical point; we can estimate the directional derivative in any direction and see that it is not zero, so the gradient is not 0. Point B is a typical critical point; from the contour values, we can see that it is a local maximum. Point C is a typical saddle point: it is at the intersection of two contours, therefore the directional derivatives in the direction of each contour is 0; since two different directional derivatives are 0, the gradient must be 0; moreover, we can see from the contour values that the values increase as we move horizontally and decrease as we move vertically, leading to a saddle-shaped graph.
11. (5, p.709) If $f(x,y) = x^3 - 3x + y^3 - 3y$, $0 = Df = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 3x^2 - 3 & 3y^2 - 3 \end{bmatrix}$ implies that $0 = 3x^2 - 3 = 3(x-1)(x+1)$ and so $x = \pm 1$. Likewise, $y = \pm 1$. In addition, $D^2 f = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$, so its discriminant is $\begin{vmatrix} 6x & 0 \\ 0 & 6y \end{vmatrix} = 36xy$. At $(x,y) = (-1,1)$ or $(1,-1)$, this is negative, so (by the Theorem on p. 708) we have saddle points. Since this is positive at the other two critical points, they are local max/min. At $(1,1)$, $f_{xx} = 6 > 0$, so we have a local min; at $(-1,-1)$, $f_{xx} = -6 < 0$, which indicates a local max.
12. (21, p.709) At $(1,3)$, $Df = \begin{bmatrix} f_x & f_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} = 0$, so it is a critical point. The discriminant is $\left|D^2 f\right| = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx} = f_{xx} f_{yy} - 0 \cdot 0 > 0$, so we have a local max/min; since $f_{xx} > 0$, so we have a local min. Here is a possible contour plot:



13. (3, p. 716) $f(x,y) = -2x^2 - 7y^2 = -(2x^2 + 7y^2)$, since $x^2, y^2 \geq 0$, $f \leq 0$. Since $f(0,0) = 0$, 0 is both an upper-bound and a function value, it is a global maximum.

14. (5, p. 716) A contour graph of $f\left(x,y\right)=x^2+y^2$:



shows that (0, 0) is a local max/min; since the contours are increasing as we move away from the origin, this is a local min. Since we can see the entire domain, (0, 0) is a global min on the unit square. Likewise, we can see that the highest contours occur at the corners; by symmetry, they all have the same value, $f\left(\pm1,\pm1\right)=\left(\pm1\right)^2+\left(\pm1\right)^2=2$. Thus, the points (1, 1), (-1, 1), (1, -1), (-1, -1) are all global maxima.

15. (21, p. 717) If $R=f\left(t,h\right)=27800-5t^2-6ht-3h^2+400t+300h$, $0=Df=\left[\,f_t \quad f_h\,\right]=\left[-10t-6h+400 \quad -6t-6h+300\right]$ implies that:

$$\begin{matrix}10t+6h=400\\6t+6h=300\end{matrix}\Rightarrow\begin{matrix}5t+3h=200\\t+h=50\end{matrix}\Rightarrow\begin{matrix}2t\quad=50\\t+h=50\end{matrix}\Rightarrow\begin{matrix}t=25\\h=25\end{matrix}$$

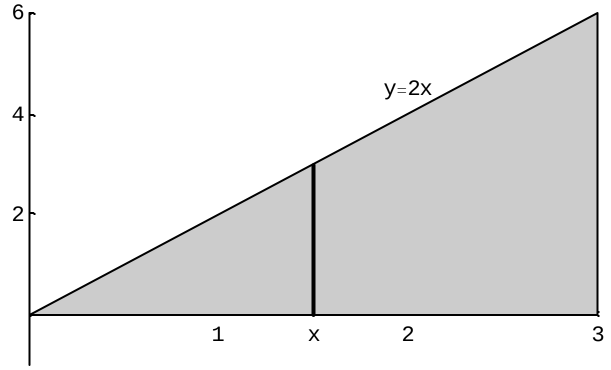
Note: At the 2nd step, we subtracted 3 times the 2nd equation from the first. In addition, $\left|D^2f\right|=\begin{vmatrix}f_{tt} & f_{ht}\\ f_{th} & f_{hh}\end{vmatrix}=\begin{vmatrix}-10 & -6\\ -6 & -6\end{vmatrix}=48>0$, so we have a local max/min. Since $f_{xx}=-10<0$, we have a local max. From the application, we expect that this is a global maximum over the domain of applicability.

16. (9, p. 742) From the Theorem on p. 740, if the temperature at a point $\left(x,y\right)$ is given by a function, $f\left(x,y\right)$, the average temperature is $\frac{1}{\text{Area}\left(R\right)}\int_R f\left(x,y\right)dA$. First, $\text{Area}\left(R\right)=25$. If we use the contour plot to estimate function values in the center of each square, we can estimate a Riemann sum for $\int_R f\left(x,y\right)dA$; working from left-to-right and top-to-bottom, we obtain the following approximate function values: 24, 23, 22, 22.5, 23.5, 25, 24.5, 24, 24, 24, 26, 25.5, 25, 25, 25, 27.5, 27, 26, 26, 25.5, 29, 28, 27, 26.5, 26. Totally these up, multiplied by the area of each rectangle (which is just 1), this gives a Riemann sum of 631.5, giving an average value of $\frac{631.5}{25}\approx25.3$.

Note: This is only an approximate answer; if you choose your points differently, you may get a slightly different result; for example, the solution in the book is 25.2.

17. (17, p. 743) The graph of $f(x,y)=5x$ is symmetric with respect to the x-axis; the graph over $x > 0$ is identical and opposite in sign to the graph over $x < 0$. Since B is also symmetric with respect to the x-axis, the integral over B with $x > 0$ is identical and opposite in sign to the integral over $x < 0$. Therefore, $\int_B 5x\,dA = 0$.

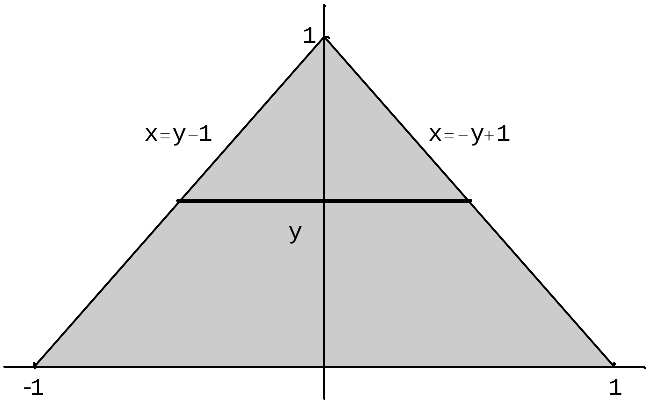
18. (15, p. 750) Considering the integral $\int_0^3 \int_0^{2x} \left(x^2+y^2\right) dy\,dx$, we can identify the bounding curves as $x=3$, $x=0$, $y=2x$, and $y=0$. Practically speaking, for each x-value between 0 and 3, we are summing functional values as y ranges from 0 to $2x$, which we can visualize as:



Simplifying gives:

$$\int_0^3 \int_0^{2x} \left(x^2+y^2\right) dy\,dx = \int_0^3 x^2 y + \frac{y^3}{3}\bigg|_0^{2x} dx = \int_0^3 2x^3 + \frac{(2x)^3}{3} dx = \frac{14}{3}\frac{x^4}{4}\bigg|_0^3 = \frac{14}{3}\frac{3^4}{4} = 94.5.$$

19. (21, p. 750) The region is bounded by the curves $y=x+1$, $y=-x+1$, and $y=0$. If we "slice" this region horizontally, that is, for each y-value between 0 and 1, we can sum functional values as x ranges from y - 1 to -y + 1 (Note: We solved the bounding equations for x), which we can visualize as:



Simplifying the integral gives:

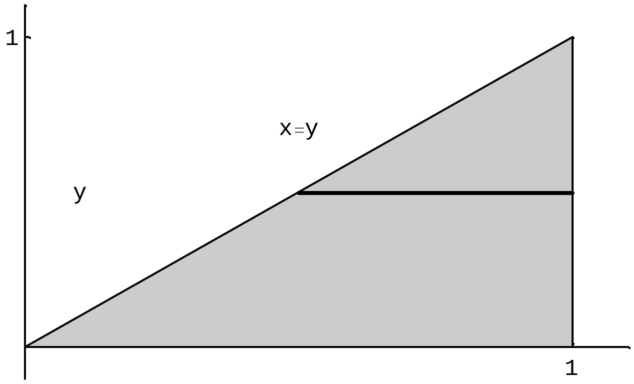
$$\int_0^1\int_{y-1}^{-y+1}(2x+3y)^2\,dxdy=\int_0^1\frac{1}{2}\frac{(2x+3y)^3}{3}\Bigg|_{y-1}^{-y+1}dx$$

$$=\frac{1}{6}\int_0^1\left(2(-y+1)+3y\right)^3-\left(2(y-1)+3y\right)^3dx$$

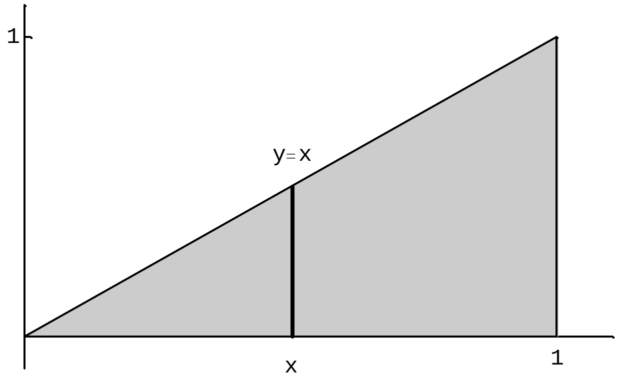
$$=\frac{1}{6}\int_0^1(y+2)^3-(5y-2)^3dx=\frac{1}{6}\left(\frac{(y+2)^4}{4}-\frac{1}{5}\frac{(5y-2)^4}{4}\right)\Bigg|_0^1$$

$$=\frac{1}{6}\left[\left(\frac{(1+2)^4}{4}-\frac{1}{5}\frac{(5\cdot1-2)^4}{4}\right)-\left(\frac{(0+2)^4}{4}-\frac{1}{5}\frac{(5\cdot0-2)^4}{4}\right)\right]=\frac{13}{6}$$

20. (27, p. 750) The region is bounded by the curves $y=0$, $y=1$, $x=y$, and $x=1$. We can view this region as "sliced" horizontally, that is, for each *y*-value between 0 and 1, we can sum functional values as *x* ranges from y to 1, which we can visualize as:



If we instead "slice" vertically, so that for each *x*-value between 0 and 1, we sum functional values as *y* ranges from 0 to *x*, this is visualized as:



Expressed as an integral, this is $\int_0^1\int_0^x\sin\left(x^2\right)dydx$, which simplifies as:

$$\int_0^1\int_0^x\sin\left(x^2\right)dydx=\int_0^1\sin\left(x^2\right)y\Big|_0^x dx=\int_0^1\sin\left(x^2\right)xdx$$

$$=-\frac{1}{2}\cos\left(x^2\right)\Big|_0^1=\frac{1}{2}\left[1-\cos(1)\right]$$

21. (3, p. 755) This may be written as $\int_0^1\int_0^1\int_0^2 (ax+by+cz)\,dzdydx$, which simplifies as:

$$\int_0^1\int_0^1\int_0^2 (ax+by+cz)\,dzdydx = \int_0^1\int_0^1 \left(axz+byz+c\frac{z^2}{2} \right)\Bigg|_0^2 dydx$$

$$= \int_0^1\int_0^1 (2ax+2by+2c)\,dydx = \int_0^1 \left(2axy+2b\frac{y^2}{2}+2cy \right)\Bigg|_0^1 dx$$

$$= \int_0^1 (2ax+b+2c)\,dx = \left(ax^2+bx+2cx\right)\Big|_0^1 = a+b+2c$$

22. (17, p. 755) Since the plane intersects the *xy*-plane in the line *x*/3 + *y*/2 = 1, which intersects the *x*-axis at *x* = 3, we can express the solid as bounded by the planes $x=0$ (i.e., the *yz*-plane), $x=3$, $y=0$ (i.e., the *xz*-plane), $y=-\frac{2}{3}x+2$, $z=0$ (i.e., the *xy*-plane), and $z=-2x-3y+6$. This means that the mass, which is the integral of the density, of the region may be expressed as $\int_0^3\int_0^{-\frac{2}{3}x+2}\int_0^{-2x-3y+6} (x+y)\,dzdydx$. This simplifies as:

$$\int_0^3\int_0^{-\frac{2}{3}x+2}\int_0^{-2x-3y+6} (x+y)\,dzdydx = \int_0^3\int_0^{-\frac{2}{3}x+2} (x+y)\,z\Big|_0^{-2x-3y+6} dydx$$

$$= \int_0^3\int_0^{-\frac{2}{3}x+2} (x+y)(-2x-3y+6)\,dydx$$

$$= \int_0^3\int_0^{-\frac{2}{3}x+2} \left[(6x-2x^2)+(6-5x)y-3y^2 \right] dydx$$

$$= \int_0^3 \left[(6x-2x^2)y+(6-5x)\frac{y^2}{2}-y^3 \right]\Bigg|_0^{-\frac{2}{3}x+2} dx$$

$$= \int_0^3 \left(-\frac{2}{3}x+2 \right) \left[(6x-2x^2)+(6-5x)\left(-\frac{1}{3}x+1 \right) - \left(-\frac{2}{3}x+2 \right)^2 \right] dx$$

$$= \int_0^3 \left(\frac{14}{27}x^3-\frac{8}{3}x^2+2x+4 \right) dx = \left(\frac{7}{54}x^4-\frac{8}{9}x^3+x^2+4x \right)\Bigg|_0^3 = 7.5$$