6.1 INNER PRODUCT, LENGTH, ORTHOGONALITY, & ANGLE

- If **u** and **v** are vectors in $\mathbb{R}^n$, then we regard **u** and **v** as n x 1 matrices.

- The transpose $\mathbf{u}^T$ is a 1xn matrix, and the matrix product $\mathbf{u}^T\mathbf{v}$ is a 1 x 1 matrix, which we write as a single real number (a scalar) without brackets.

- The number $\mathbf{u}^T\mathbf{v}$ is called the **inner product (or dot product)** of **u** and **v**, and it is written as $u \cdot v$.

- If $\mathrm{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathrm{v}=\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then the inner product of **u** and **v** is

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

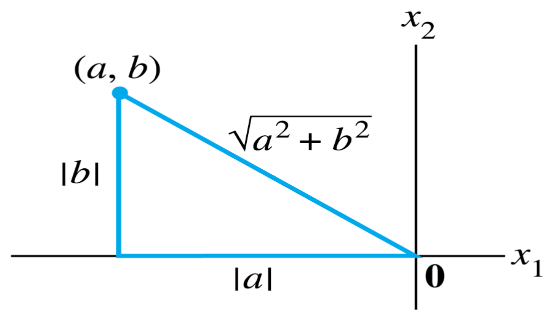
Example: Let $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ find inner product between u and v.

If **v** is in $\mathbb{R}^n$, with entries $v_1, \ldots, v_n$, then the square root of $v \cdot v$ is defined because $v \cdot v$ is nonnegative.

Definition: The **length** (or **norm**) of **v** is the nonnegative scalar $\|v\|$ defined by

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } \|v\|^2 = v \cdot v$$

- If we identify **v** with a geometric point in the plane, as usual, then $\|\mathrm{v}\|$ coincides with the standard notion of the length of the line segment from the origin to **v**.
- This follows from the Pythagorean Theorem applied to a triangle such as the one shown in the following figure.



Interpretation of $\|\mathbf{v}\|$ as length.

- For any scalar *c*, the length *c***v** is $|c|$ times the length of v. That is,

$$\|c\mathrm{v}\| = |c|\|\mathrm{v}\|$$

A vector whose length is 1 is called a **unit vector**.

If we *divide* a nonzero vector **v** by its length—that is, multiply by $1/\|\mathrm{v}\|$—we obtain a unit vector **u** because the length of **u** is $(1/\|\mathrm{v}\|)\|\mathrm{v}\|$

- **Example:** Let $\mathrm{v} = (1, -2, 2, 0)$. Find a unit vector **u** in the same direction as **v**.
- **Solution:** First, compute the length of **v**:

$$\|\mathrm{v}\|^2 = \mathrm{v} \bullet \mathrm{v} = (1)^2 + (-2)^2 + (2)^2 + (0)^2 = 9$$

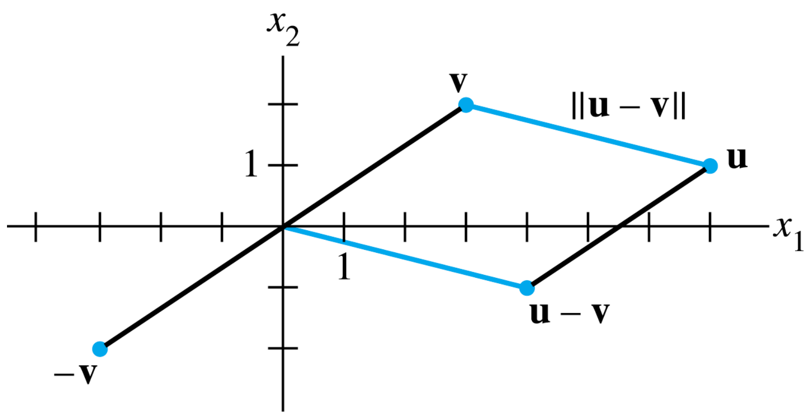
$$\|\mathrm{v}\| = \sqrt{9} = 3$$

- Then, multiply **v** by $1/\|\mathrm{v}\|$ to obtain

$$\mathrm{u} = \frac{1}{\|\mathrm{v}\|}\mathrm{v} = \frac{1}{3}\mathrm{v} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

- **Definition:** For **u** and **v** in $\mathbb{R}^n$, the **distance between u and v**, written as dist (**u**, **v**), is the length of the vector $\mathbf{u} - \mathbf{v}$

. That is, $\text{dist}\,(\mathrm{u},\mathrm{v}) = \|\mathrm{u} - \mathrm{v}\|$



The distance between **u** and **v** is the length of **u** − **v**.

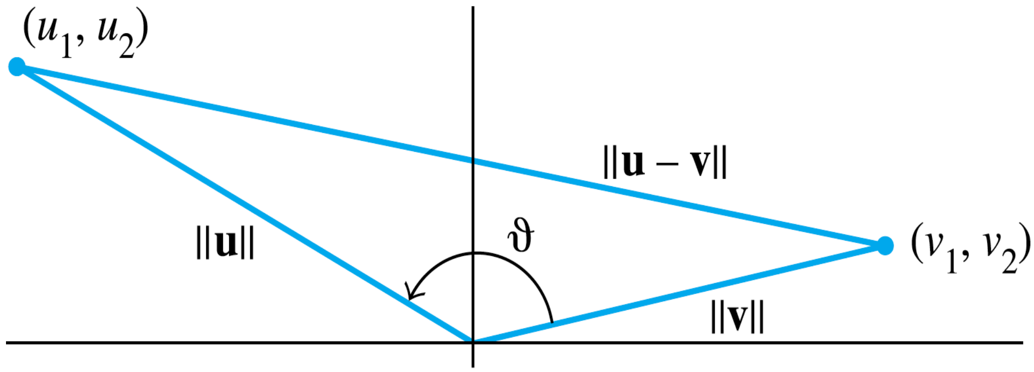
Example: Compute the distance between the vectors $\mathrm{u} = (7,1)$ and $\mathrm{v} = (3,2)$

$$\mathrm{u} - \mathrm{v} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\|\mathrm{u} - \mathrm{v}\| = \sqrt{4^2 + (-1)^2} = \sqrt{17}$$

- If **u** and **v** are nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, then there is a nice connection between their inner product and the angle ϑ between the two line segments from the origin to the points identified with **u** and **v**.

- The formula is $\mathrm{u} \bullet \mathrm{v} = \|\mathrm{u}\|\|\mathrm{v}\|\cos\vartheta$



The angle between two vectors.

Law of cosines,

$$\|\mathrm{u} - \mathrm{v}\|^2 = \|\mathrm{u}\|^2 + \|\mathrm{v}\|^2 - 2\|\mathrm{u}\|\|\mathrm{v}\|\cos\vartheta$$

Example: Let $u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$ find angle between u and v.

- **Definition:** Two vectors **u** and **v** in $\mathbb{R}^n$ are **orthogonal** (to each other) if $u \cdot v = 0$.