# Section 5.2 The Characteristic Equation

**Review:**

$$A \quad \mathbf{x} \quad = \quad \lambda \quad \mathbf{x}$$

Find eigenvectors **x** by solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

**How do we find the eigenvalues** $\lambda$?

**x** must be nonzero

$$\Downarrow$$

$(A - \lambda I)\mathbf{x} = \mathbf{0}$ must have nontrivial solutions

$$\Downarrow$$

$(A - \lambda I)$ is not invertible

$$\Downarrow$$

$$\det(A - \lambda I) = 0$$

(called the *characteristic equation*)

| Solve $\det(A - \lambda I) = 0$ for $\lambda$ to find the eigenvalues. |
|---|

*Characteristic polynomial:* $\det(A - \lambda I)$

*Characteristic equation:* $\det(A - \lambda I) = 0$

**EXAMPLE:** Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

*Solution:* Since

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{bmatrix},$$

the equation $\det(A - \lambda I) = 0$ becomes

$$-\lambda(5 - \lambda) + 6 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

Factor:

$$(\lambda - 2)(\lambda - 3) = 0.$$

So the eigenvalues are $2$ and $3$.

For a $3 \times 3$ matrix or larger, recall that a determinant can be computed by cofactor expansion.

**EXAMPLE:** Find the eigenvalues of $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & 0 \\ 1 & 8 & 1 \end{bmatrix}$.

*Solution:*

$$A - \lambda I = \begin{bmatrix} 1 - \_\_\_\_ & 2 & 1 \\ 0 & -5 - \_\_\_\_ & 0 \\ 1 & 8 & 1 - \_\_\_\_ \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & -5-\lambda & 0 \\ 1 & 8 & 1-\lambda \end{vmatrix} = (-5-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (-5-\lambda)\left[(1-\lambda)^2 - 1\right] = (-5-\lambda)[1 - 2\lambda + \lambda^2 - 1]$$

$$= (-5-\lambda)[-2\lambda + \lambda^2] = -(5+\lambda)\lambda[-2+\lambda] = 0$$

$$\Rightarrow \lambda = -5, 0, 2$$

**THEOREM** (The Invertible Matrix Theorem - continued)

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if:

s. The number $0$ is not an eigenvalue of $A$.

t. $\det A \neq 0$

Recall that if $B$ is obtained from $A$ by a sequence of row replacements or interchanges, but without scaling, then $\det A = (-1)^r \det B$, where $r$ is the number of row interchanges.

Suppose the echelon form $U$ is obtained from $A$ by a sequence of row replacements or interchanges, but without scaling.

$$A \sim U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & u_{nn} \end{bmatrix}$$

The **determinant** of $A$, written $\det A$, is defined as follows:

$$\det A = \begin{cases} (-1)^r \cdot \left( \begin{array}{c} \text{product of} \\ \text{pivots in } U \end{array} \right), & \text{when } A \text{ is invertible} \\ 0, & \text{when } A \text{ is not invertible} \end{cases}$$

($r$ is the number of row interchanges)

**EXAMPLE:** Find the eigenvalues of $A = \begin{bmatrix} 3 & 2 & 3 \\ 0 & 6 & 10 \\ 0 & 0 & 2 \end{bmatrix}$.

*Solution:*

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 2 & 3 \\ 0 & 6-\lambda & 10 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Characteristic equation:

$$(\qquad)(\qquad)(\qquad) = 0.$$

**eigenvalues:** _____, _____, _____

The (**algebraic**) **multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.

**EXAMPLE:** Find the characteristic polynomial of

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 9 & 1 & 3 & 0 \\ 1 & 2 & 5 & -1 \end{bmatrix}$$

and then find all the eigenvalues and the algebraic multiplicity of each eigenvalue.

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 & 0 \\ 5 & 3-\lambda & 0 & 0 \\ 9 & 1 & 3-\lambda & 0 \\ 1 & 2 & 5 & -1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda)(3-\lambda)(-1-\lambda) = 0$$

**eigenvalues:** _____, _____, _____

**Similarity**

Numerical methods for finding approximating eigenvalues are based upon Theorem 4 to be described shortly.

For $n \times n$ matrices $A$ and $B$, we say the $A$ is **similar** to $B$ if there is an invertible matrix $P$ such that

$$P^{-1}AP = B \qquad \text{or equivalently,} \qquad A = PBP^{-1}.$$

**Theorem 4:** If $n \times n$ matrices $A$ and $B$ are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

**Proof:** If $B = P^{-1}AP$, then

$$\det(B - \lambda I) = \det[P^{-1}AP - P^{-1}\lambda IP] = \det[P^{-1}(A - \lambda I)P]$$

$$= \det P^{-1} \cdot \det(A - \lambda I) \cdot \det P = \det(A - \lambda I).$$