MATH 215
Practice Section 2.1

Given the matrices below $A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

1. Compute the product AB in two ways:

(a) by the definition Ab_1 and Ab_2

$$\overset{A}{\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}} \cdot \begin{array}{cc} b_1 & b_2 \\ \end{array}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

$$= \left[\underbrace{1 \cdot \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}}_{Ab_1} \quad \underbrace{3 \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} - 1 \cdot \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}}_{Ab_2} \right]$$

$$= \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$$

(b) the row-column rule.

$$\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1(4) + 2(-2) & 3(4) + (-1)(-2) \\ 1 \cdot (-3) + 2(0) & 3(-3) + (-1)(0) \\ 1 \cdot (3) + 2(5) & 3(3) + (-1)(5) \end{bmatrix} = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}$$

2. Compute $(AB)^T$, A^T, B^T, $A^T B^T$ and $B^T A^T$.

$$(AB)^T = \begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}^T = \begin{bmatrix} 0 & -3 & 13 \\ 14 & -9 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -3 & 3 \\ -2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 13 \\ 14 & -9 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 & 3 \\ -2 & 0 & 5 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$$A^T B^T = \underbrace{\begin{bmatrix} 4 & -3 & 3 \\ -2 & 0 & 5 \end{bmatrix}}_{2 \times 3} \cdot \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}}_{2 \times 2}$$ = Cannot be done

undefined.

3. Let $u = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Compute $u^T u$, uu^T, $u^T v$, $v^T u$, vu^T, and uv^T.

$$\boxed{u^T u = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 3 \cdot 3 + (-2)(-2) + 1 \cdot 1 = 14}$$

$$\boxed{uv^T = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 & 0\cdot 3 & 1\cdot 3 \\ 2\cdot(-2) & 0\cdot(-2) & 1\cdot(-2) \\ 2\cdot(1) & 0\cdot(1) & 1\cdot(1) \end{bmatrix} = \begin{bmatrix} 6 & 0 & 3 \\ -4 & 0 & -2 \\ 2 & 0 & 1 \end{bmatrix}}$$

$$\boxed{uu^T = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 3 & -2\,(3) & 1\cdot(3) \\ 3\cdot(-2) & -2\,(-2) & 1\cdot(-2) \\ 3\cdot(1) & -2\,(1) & 1\cdot(1) \end{bmatrix} = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 4 & -2 \\ 3 & -2 & 1 \end{bmatrix}}$$

$$\boxed{u^T v = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 3\cdot 2 + 0\cdot(-2) + 1\cdot 1 = 7}$$

$$\boxed{v^T u = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = 3\cdot 2 + 0\cdot(-2) + (1)\cdot(1) = 7}$$

$$\boxed{vu^T = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2 & -2\cdot 2 & 1\cdot 2 \\ 3\cdot 0 & -2\cdot 0 & 1\cdot 0 \\ 3\cdot 1 & -2\cdot 1 & 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 2 \\ 0 & 0 & 0 \\ 3 & -2 & 1 \end{bmatrix}}$$

4. Let $u = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Compute $P = I - 2uu^T/(u^T u)$ where I is the 2×2 identity matrix. What does $P^2 =$? After finding P^2 for the specific example, try to show this result in general. i.e. what does $P^2 =$? for any vector u in R^n.

$$u^T u = \begin{bmatrix} -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -2\cdot(-2) + 1\cdot 1 = 5$$

$$uu^T = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & 1 \end{bmatrix} = \begin{bmatrix} -2\cdot(-2) & 1\,(-2) \\ -2\cdot(1) & 1\cdot(1) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} / 5 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2/5 \cdot \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

$$P^2 = P \cdot P = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \cdot \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} P^2 = P \cdot P &= \left(I - \tfrac{2}{u^T u} \cdot uu^T\right)\left(I - \tfrac{2}{u^T u} \cdot uu^T\right) = I - \tfrac{2}{u^T u} \cdot uu^T - \tfrac{2}{u^T u} \cdot uu^T + \tfrac{4}{(u^T u)^2} \cdot u(u^T u)u^T \\ &= I - \tfrac{4}{u^T u} uu^T + \tfrac{4}{u^T u} \cdot uu^T \\ &= I. \end{aligned}$$