

MATH 215
Practice -Exam 2

1. Find the inverse of the given matrices. Show ALL row operations that you used.

a) $A = \begin{bmatrix} 4 & -3 \\ 8 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 8 & -1 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 4 & 0 & 1/5 & 3/5 \\ 0 & 1 & -2/5 & 1/5 \end{array} \right]$$

$1/4 R_1 \rightarrow R_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1/20 & 3/20 \\ 0 & 1 & -2/5 & 1/5 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1/20 & 3/20 \\ -2/5 & 1/5 \end{bmatrix}$$

Handwritten steps for part a:
 $-2R_1 + R_2 \rightarrow R_2$
 $1/5 R_2 \rightarrow R_2$
 $3R_2 + R_1 \rightarrow R_1$

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & 0 & -1/2 \\ 0 & -1 & 0 & -2 & 1 & -1/2 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array} \right]$$

$-3R_1 + R_2 \rightarrow R_2$
 $-2R_1 + R_3 \rightarrow R_3$
 $1/2 R_3 \rightarrow R_3$
 $-R_3 + R_2 \rightarrow R_2$
 $-R_3 + R_1 \rightarrow R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/4 & -7/8 \\ 0 & 1 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 0 & 1 & -1/2 & -1/4 & 5/8 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & 3/4 & -7/8 \\ 1/2 & -1/4 & 1/8 \\ -1/2 & -1/4 & 5/8 \end{bmatrix}$$

c) Using the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$ from part b) above solve $Ax = b$, where $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$$Ax = b$$

$$X = A^{-1}b$$

$$X = \begin{bmatrix} 1/2 & 3/4 & -7/8 \\ 1/2 & -1/4 & 1/8 \\ -1/2 & -1/4 & 5/8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 - 14/8 \\ 1/2 + 2/8 \\ -1/2 + 10/8 \end{bmatrix} = \begin{bmatrix} -5/4 \\ 3/4 \\ 3/4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix}$$

2. Using **cofactor expansion across the first row** to compute the determinant of A . (Show all of your work!)

$$\begin{aligned} \det(A) &= (-1)^{11} \cdot 1 \cdot \det \begin{bmatrix} -3 & -5 \\ 1 & -1 \end{bmatrix} + (-1)^{12} \cdot 2 \cdot \det \begin{bmatrix} -2 & -5 \\ 2 & -1 \end{bmatrix} + (-1)^{13} \cdot 4 \cdot \det \begin{bmatrix} -2 & -3 \\ 2 & 1 \end{bmatrix} \\ &= (3+5) - 2 \cdot (2+10) + 4 \cdot (-2+6) \\ &= 8 - 24 + 16 \\ &= 0 \end{aligned}$$

3. Using **cofactor expansion down last column** to compute the determinant of B . (Show all of your work!)

$$\begin{aligned} \det(B) &= (-1)^{13} \cdot (-2) \cdot \det \begin{bmatrix} 2 & 7 \\ 2 & 9 \end{bmatrix} + (-1)^{23} \cdot (-1) \cdot \det \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} + (-1)^{33} \cdot (7) \cdot \det \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \\ &= -2 \cdot (18-14) + (9-8) + 7 \cdot (7-8) \\ &= -8 + 1 - 7 \\ &= -14 \end{aligned}$$

4. Using row operations combined with cofactor expansion, compute the determinant of A and the determinant of B .

$$\begin{aligned} \det(A) &= \det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix} \xrightarrow{\substack{2R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3}} \det \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = 0 \\ \det(B) &= \det \begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & 11 \end{bmatrix} \xrightarrow{\substack{-2R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3}} \det \begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \xrightarrow{R_2 + R_3 \rightarrow R_2} = 1 \cdot (-1) \cdot 14 = -14. \end{aligned}$$

5. Compute the product AB . What is the determinant of AB ? What is the determinant of A^T ?

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 13 & 54 & 24 \\ -18 & -74 & -28 \\ 2 & 6 & -12 \end{bmatrix} \quad \det(A^T) = \det(A) = 0 \\ \det(AB) &= \det(A) \cdot \det(B) = 0 \cdot (-14) = 0 \end{aligned}$$

6. Matrix A invertible? Matrix B invertible? Do the columns of A span \mathbf{R}^3 ?
Are the columns of B linearly independent?

A is singular since $\det(A) = 0$

B is non-singular (has an inverse) since $\det(B) \neq 0$

Inv. matrix th. columns of A do not span \mathbf{R}^3

Inv. matrix th. columns of B are linearly independent.