MATH 215
Practice -Exam 2

1. Find the inverse of the given matrices. Show **ALL** row operations that you used.

a) $A = \begin{bmatrix} 4 & -3 \\ 8 & -1 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 8 & -1 & 0 & 1 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right]$$

$\frac{1}{5}R_2 \to R_2$

$$\left[\begin{array}{cc|cc} 4 & -3 & 1 & 0 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$3R_2 + R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 4 & 0 & -1/5 & 3/5 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$\frac{1}{4}R_1 \to R_1$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1/20 & 3/20 \\ 0 & 1 & -2/5 & 1/5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1/20 & 3/20 \\ -2/5 & 1/5 \end{bmatrix}$$

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 3 & 1 & 4 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$-3R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -3 & 1 & 0 \\ 0 & -2 & 2 & -2 & 0 & 1 \end{array}\right]$$

$\frac{1}{2}R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -3 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

$-R_3 + R_2 \to R_2$
$-R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & 0 & -1/2 \\ 0 & -4 & 0 & -2 & 1 & -1/2 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

$-\frac{1}{4}R_2 \to R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & -1 & 1 & -1 & 0 & 1/2 \end{array}\right]$$

$-3R_2 + R_1 \to R_1$
$R_2 + R_3 \to R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/4 & -7/8 \\ 0 & 1 & 0 & 1/2 & -1/4 & 1/8 \\ 0 & 0 & 1 & -1/2 & -1/4 & 5/8 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & 3/4 & -7/8 \\ 1/2 & -1/4 & 1/8 \\ -1/2 & -1/4 & 5/8 \end{bmatrix}$$

c) Using the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 2 & 2 & 4 \end{bmatrix}$ from part b) above solve $Ax = b$, where $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$Ax = b$

$X = A^{-1}b$

$$X = \begin{bmatrix} 1/2 & 3/4 & -7/8 \\ 1/2 & -1/4 & 1/8 \\ -1/2 & -1/4 & 5/8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 - 14/8 \\ 1/2 + 2/8 \\ -1/2 + 10/8 \end{bmatrix} = \begin{bmatrix} -5/4 \\ 3/4 \\ 3/4 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix}$$

2. Using **cofactor expansion across the first row** to compute the determinant of A. (Show all of your work!)

$$\det(A) = (-1)^{1+1} \cdot 1 \cdot \det\begin{pmatrix} -3 & -5 \\ 1 & -1 \end{pmatrix} + (-1)^{1+2} \cdot 2 \cdot \det\begin{bmatrix} -2 & -5 \\ 2 & -1 \end{bmatrix} + (-1)^{1+3} \cdot 4 \cdot \det\begin{bmatrix} -2 & -3 \\ 2 & 1 \end{bmatrix}$$
$$= (3+5) - 2\cdot(2+10) + 4\cdot(-2+6)$$
$$= 8 - 24 + 16$$
$$= 0$$

3. Using **cofactor expansion down last column** to compute the determinant of B. (Show all of your work!)

$$\det(B) = (-1)^{1+3} \cdot (-2) \cdot \det\begin{bmatrix} 2 & 7 \\ 2 & 9 \end{bmatrix} + (-1)^{2+3} \cdot (-1) \cdot \det\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} + (-1)^{3+3} \cdot (7) \cdot \det\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$
$$= -2\cdot(18-14) + (9-8) + 7\cdot(7-8)$$
$$= -8 + 1 - 7$$
$$= -14$$

4. Using row operations combined with cofactor expansion, compute the determinant of A and the determinant of B

$$\det(A) = \det\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & -3 & -9 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$2R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$3R_2 + R_3 \to R_3$

$$\det(B) = \det\begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 1 & 11 \end{bmatrix} = \det\begin{bmatrix} 1 & 4 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 14 \end{bmatrix}$$

$-2R_1 + R_2 \to R_2$
$-2R_1 + R_3 \to R_3$

$R_2 + R_3 \to R_3$

$$= 1\cdot(-1)\cdot 14 = -14.$$

5. Compute the product AB. What is the determinant of AB? What is the determinant of A^T?

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & -2 \\ 2 & 7 & -1 \\ 2 & 9 & 7 \end{bmatrix} = \begin{bmatrix} 13 & 54 & 24 \\ -18 & -74 & -28 \\ 2 & 6 & -12 \end{bmatrix}$$

$$\det(A^T) = \det(A) = 0.$$

$$\det(AB) = \det(A)\cdot\det(B) = 0\cdot(-14) = 0$$

6. Matrix A invertible? Matrix B invertible? Do the columns of A span $\mathbf{R}^3$? Are the columns of B linearly independent?

A is singular since $\det(A) = 0$

B is non-singular (has an inverse) since $\det(B) \neq 0$

Inv. Matrix th. columns of A do not span $\mathbb{R}^3$

Inv. matrix Th. columns of B are linearly independent.

7. Let $T : R^3 \to R^2$ be the linear transformation such that $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\-4\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\end{bmatrix}$ and

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}3\\9\end{bmatrix}.$$

Find the standard matrix representation of T and $T\left(\begin{bmatrix}3\\4\\-2\end{bmatrix}\right)$.

$$A = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 1 & 9 \end{bmatrix}$$

$$T(x) = A \cdot x = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 1 & 9 \end{bmatrix}$$

$$T\begin{pmatrix}3\\4\\-2\end{pmatrix} = A \cdot \begin{pmatrix}3\\4\\-2\end{pmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 1 & 9 \end{bmatrix} \cdot \begin{pmatrix}3\\4\\-2\end{pmatrix}$$

$$= \begin{bmatrix}-4\\-26\end{bmatrix}$$