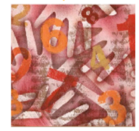
Chapter 9: Social Choice: The Impossible Dream

For All Practical Purposes



Mathematical Literacy in Today's World, 9th ed.

Section 9.3 Other Voting Systems For Three or More Candidates

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Four Desirable properties of voting systems with
3 or more candidates:

1.) Condorcet winner criterion (pg 333 9th ed)
2.) Independence of irrelevant alternatives (pg 336 9th ed)
3.) Pareto Condition (pg 338 9th ed)
4.) Monotonicity (pg 341 9th ed)

Other Voting Systems for Three or More Candidates

- Voting Systems for Three or More Candidates
  - When there are three or more candidates, it is more unlikely to have a candidate win with a majority vote.
  - Many other voting methods exist, consisting of reasonable ways to choose a winner; however, they all have shortcomings.
  - We will examine four more popular voting systems for three or more candidates:

- Four voting systems, along with their shortcomings:
  1. Plurality Voting and the Condorcet Winning Criterion
  2. The Borda Count and Independence of Irrelevant Alternatives
  3. Sequential Pairwise Voting and the Pareto Condition
  4. The Hare System and Monotonicity

Plurality Voting (Voting Procedure 1 of 4)

– Only first-place votes are considered.
  - Even if a preference list ballot is submitted, only the voters' first choice will be counted—it could have just been a single vote cast.

– The candidate with the most votes wins.
  - The winner does not need a majority of votes, but simply have more votes than the other candidates.

Example: Find the plurality vote of the 3 candidates and 13 voters.

	Number of Voters (13)			
Rank	5	4	3	1
First	**A**	**C**	**B**	**B**
Second	B	B	C	A
Third	C	A	A	C

The candidate with the most first -place votes wins. Count each candidate's first-place votes only. *(A has the most.)*

A = 5, B = 4, C = 4 **A wins.**

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Plurality Voting (Voting Procedure 1 of 4)

Example

A group of twelve students have to decide among three activities: hold a keg party (K), watch a movie (M), or study (S). Their preference rankings are shown below. Which choice will the group make if they use plurality voting?

Number of Students	3	3	2	2	2
First choice	K	M	S	K	S
Second choice	M	K	M	S	K
Third choice	S	S	K	M	M

Answer is not provide, however you should be able to solve the example.

Plurality Voting and the Condorcet Winning Criterion

- Example: 2000 Presidential Election (Plurality fails CWC.)
  - Condorcet Winner Criterion (CWC) is satisfied if either is true:
    1. If there is no Condorcet winner (often the case) - or -
    2. If the winner of the election is also the Condorcet winner
  - This election came down to which of Bush or Gore would carry Florida. Result: George W. Bush won by a few hundred votes.
  - Gore, however, was considered the Condorcet winner:

    It is assumed if Al Gore was pitted against any one of the other three candidates, (Bush, Buchanan, Nader), Gore would have won.

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Borda Count (Voting Procedure 2 of 4)

- The Borda Count
  - Borda Count is a rank method of voting that assigns points in a nonincreasing manner to the ordered candidates on each voter's preference list ballot and then add these points to arrive at a group's final ranking.
  - For n candidates, assign points as follows:
    First-place vote is worth $n - 1$ points, second-place vote is worth $n - 2$ points, and so on down to…Last place vote is worth $n - n = 0$, zero points.
  - For some problems, you can also assign points
    n, n-1, …, 1 (1 is for last place).
  - The candidate's total points are referred to as his/her Borda score.

Example: Total the Borda score of each candidate.

Rank	Number of Voters (5)					Points
First	A	A	A	B	B	2
Second	B	B	B	C	C	1
Third	C	C	C	A	A	0

$A = 2 + 2 + 2 + 0 + 0 = 6$
$B = 1 + 1 + 1 + 2 + 2 = 7$
$C = 0 + 0 + 0 + 1 + 1 = 2$

B has the most, **B wins.**

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Borda Count (Voting Procedure 2 of 4) Example

100 members of the University Marching Band are trying to decide in which of 4 different bowl games they will march. the preference schedule is given:

R: Rose Bowl **H:** Hula Bowl
C: Cotton Bowl **O:** Orange Bowl

# of votes	49	48	3
1st	R	H	C
2nd	H	O	H
3rd	C	C	O
4th	O	R	R

In which bowl will the University Band March if votes are counted by the Borda Count method? (use a 4, 3, 2, 1 point distribution).

R=49(4)+48(1)+3(1)=247
H=48(4)+49(3)+3(3)=348
C=3(4)+49(2)+48(2)=206
O=48(3)+3(2)+49(1)=199

Borda Count (Voting Procedure 2 of 4) Example

A fourteen-person committee is considering three applicants, A, B, and C, for the new Provost. The individual rankings are summarized in the table below. (No one preferred the rankings BAC or CBA.) Which applicant would be accepted if the committee used the 3-2-1 Borda count?

Number of Members	5	4	3	2
First choice	A	C	B	A
Second choice	C	A	C	B
Third choice	B	B	A	C

Answer:

$A=5(3)+4(2)+3(1)+2(3)=32$
$B=5(1)+4(1)+3(3)+2(2)=22$
$C=5(2)+4(3)+3(2)+2(1)=30$

Try doing the problem without looking at the answer.

Borda Count and Independence of Irrelevant Alternatives

- Independence of Irrelevant Alternatives (Borda fails IIA.)
  - A voting system is said to satisfy independence of irrelevant alternatives (IIA) if it is impossible for candidate B to move from nonwinner status to winner status unless at least one voter reverses the order in which he or she had B and the winning candidate ranked.
    - If B was a loser, B should never become a winner, unless he moves ahead of the winner (reverses order) in a voter's preference list.

Example showing that Borda count fails to satisfy IIA:

Original Borda Score: **A=6,** B=5, C=4

	Rank	Number of Voters (5)				
2	First	A	A	A	**C**	**C**
1	Second	B	B	B	**B**	**B**
0	Third	C	C	C	A	A

Suppose the last two voters change their ballots (reverse the order of just the losers). This should not change the winner.

B went from loser to winner and did not switch with A!

New Borda Score: A= 6, **B=7,** C=2

Rank	Number of Voters (5)				
First	A	A	A	**B**	**B**
Second	B	B	B	**C**	**C**
Third	C	C	C	A	A

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Sequential Pairwise Voting (Voting Procedure 3 of 4)

- Sequential Pairwise Voting
  - Sequential pairwise voting starts with an **agenda** and pits the first candidate against the second in a one-on-one contest.
  - The losers are deleted and the winner then moves on to confront the third candidate in the list, one on one.
  - This process continues throughout the entire **agenda**, and the one remaining at the end wins.

Example: Who would be the winner using the **agenda A, B, C, D** for the following preference list ballots of three voters?

Rank	Number of Voters (3)		
First	A	C	B
Second	B	A	D
Third	D	B	C
Fourth	C	D	A

Using the agenda A, B, C, D, start with A vs. B and record (with tally marks) who is preferred for each ballot list (column).

A vs. B	A vs. C	C vs. D
II I	I II	I II
A wins; B is deleted.	C wins; A is deleted.	D wins; C is deleted.

Candidate D wins for this agenda.

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Different agenda can produce different winners!

Sequential Pairwise Voting (Voting Procedure 3 of 4) Example

Given the agenda: B, C, D, A and the preference schedule in the following figure, who will win the election using sequential pairwise voting?

# of votes	5	2	4
1st	A	B	C
2nd	B	C	D
3rd	D	A	A
4th	C	D	B

By the given agenda, B competes first against C.

B vs C: B get 7 votes.
C get 4 votes
B wins; C is eliminated.

B goes on to compete with the next alternative, D

B vs D: B gets 7 votes
D gets 4 votes
B wins; D is eliminated

Winner is A!!

B vs A: B gets 2 votes
A gets 9 votes
A wins; B is eliminated.

Sequential Pairwise Voting and the Pareto Condition

- Pareto Condition (Sequential Pairwise fails Pareto.)
  - Pareto condition states that if everyone prefers one candidate (in this case, B) to another candidate (D), then this latter candidate (D) should not be among the winners of the election.
  - Pareto condition is named after Vilfredo Pareto (1848 – 1923), Italian economist.

Example:

- D was the winner for the agenda A, B, C, D.
- However, each voter (each of the three preference lists columns) preferred B over D.
- If everyone preferred B to D, then D should not have been the winner! *Not fair!*

Rank	**Number of Voters (3)**		
First	A	C	**B**
Second	**B**	A	**D**
Third	**D**	**B**	C
Fourth	C	**D**	A

Different agenda orders can change the outcomes. For example, agenda D, C, B, A results in A as the winner.

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The Hare System (Voting Procedure 4 of 4)

- The Hare System
  - The Hare system proceeds to arrive at a winner by repeatedly deleting candidates that are "least preferred" (meaning at the top of the fewest ballots).
  - If a single candidate remains after all others have been eliminated, he/she alone is the winner.
  - If two or more candidates remain and they all would be eliminated in the next round, then these candidates would tie.

	Number of Voters (13)			
Rank	**5**	**4**	**3**	**1**
First	**A**	**C**	**B**	**B**
Second	B	B	C	A
Third	C	A	A	C

For the Hare system, delete the candidate with the least first -place votes:

A = 5, B = 4, and C = 4

Since B and C are tied for the least first place votes, they are both deleted and **A wins.**

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The Hare System (Voting Procedure 4 of 4) Example

Example: Using the preference schedule in the following figure, which candidate will win if the Hare System of voting is used?

# of votes	7	5	4	1
1st	A	C	B	D
2nd	D	A	C	A
3rd	B	B	D	B
4th	C	D	A	C

Step 1. D has fewest 1st place votes → D is eliminated. Remove D from chart and move others up.

# of Votes	7	5	4	1
1st	A	C	B	A
2nd	B	A	C	B
3rd	C	B	A	C

Step 2. B now has the fewest 1st place votes → B is eliminated. Remove B from lists and move others up.

# of votes	7	5	4	1
1st	A	C	C	A
2nd	C	A	A	C

Step 3. A now has fewest 1st place votes and is eliminated!

C wins!!

The Hare System and Monotonicity

Monotonicity (The Hare system fails monotonicity.)

- Monotonicity says that if a candidate is a winner and a new election is held in which the only ballot change made is for some voter to move the former winning candidate higher on his or her ballot, then the original winner should remain a winner.
- In a new election, if a voter moves a winner higher up on his preference list, the outcome should still have the same winner.

	Number of Voters (13)			
Rank	**5**	**4**	**3**	**1**
First	A	C	B	**A**
Second	B	B	C	**B**
Third	C	A	A	C

In the previous example, A won. For the last voter, move A up higher on the list (A and B switch places). Round 1: B is deleted. Round 2: C moves up to replace B on the third column. However, C wins—this is a glaring defect!

– The Hare system, introduced by Thomas Hare in 1861, was known by names such as the "single transferable vote system." In 1962, John Stuart Mill described the Hare system as being "among the greatest improvements yet made in the theory and practice of government."

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