

2.2 The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1} . For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

FACT If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A . Then

$$B = BI = B(\text{_____}) = (\text{_____})\text{_____} = I\text{_____} = C.$$

So the inverse is unique since any two inverses coincide. ■

The inverse of A is usually denoted by A^{-1} .

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ matrices are invertible. A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$$\text{_____} A\mathbf{x} = \text{_____} \mathbf{b} \quad \text{and so}$$

$$I\mathbf{x} = \text{_____} \quad \text{or } \mathbf{x} = \text{_____}.$$

Suppose \mathbf{w} is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$$\text{_____} A\mathbf{w} = \text{_____} \mathbf{b} \quad \text{which means } \mathbf{w} = A^{-1}\mathbf{b}.$$

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Theorem 5

If A is an invertible $n \times n$ matrix, then for each \mathbf{b} in \mathbf{R}^n , the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve $\begin{cases} -7x_1 + 3x_2 = 2 \\ 5x_1 - 2x_2 = 1 \end{cases}$.

Solution: Matrix form of the linear system: $\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{14-15} \begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

Theorem 6 Suppose A and B are invertible. Then the following results hold:

- A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(\text{_____})A^{-1} \\ &= A(\text{_____})A^{-1} = \text{_____} = \text{_____}. \end{aligned}$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \text{_____}$$

Earlier, we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary** matrices.

Elementary Matrices

Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

E_1 , E_2 , and E_3 are elementary matrices. Why?

Observe the following products and describe how these products can be obtained by elementary row operations on A .

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A , the resulting matrix can be written as EA , where the $m \times m$ matrix E is created by performing the same row operations on I_m .

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E , determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2E_1A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\boxed{E_3E_2E_1A = I_3}.$$

Then multiplying on the right by A^{-1} , we get

$$E_3E_2E_1A \underline{\hspace{1cm}} = I_3 \underline{\hspace{1cm}}.$$

So

$$\boxed{E_3E_2E_1I_3 = A^{-1}}$$

The elementary row operations that row reduce A to I_n are the same elementary row operations that transform I_n into A^{-1} .

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n , and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1} .

Algorithm for finding A^{-1}

Place A and I side-by-side to form an augmented matrix $[A \ I]$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem 7:

$$[A \ I] \text{ will row reduce to } [I \ A^{-1}]$$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$[A \ I] = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

$$\text{So } A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$$

Order of multiplication is important!

EXAMPLE Suppose A, B, C , and D are invertible $n \times n$ matrices and $A = B(D - I_n)C$.

Solve for D in terms of A, B, C and D .

Solution:

$$\underline{\hspace{2cm}} A \underline{\hspace{2cm}} = \underline{\hspace{2cm}} B(D - I_n)C \underline{\hspace{2cm}}$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \underline{\hspace{2cm}} = B^{-1}AC^{-1} + \underline{\hspace{2cm}}$$

$$D = \underline{\hspace{4cm}}$$