

1.3 VECTOR EQUATIONS

Key concepts to master: linear combinations of vectors and a spanning set.

Vector: A matrix with only one column.

Vectors in \mathbb{R}^n (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of \mathbb{R}^2

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

\mathbb{R}^2 is the set of all points in the plane.

Parallelogram rule for addition of two vectors:

If \mathbf{u} and \mathbf{v} in \mathbb{R}^2 are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, \mathbf{u} and \mathbf{v} . (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of \mathbf{u}, \mathbf{v} and $\mathbf{u} + \mathbf{v}$ are given below:

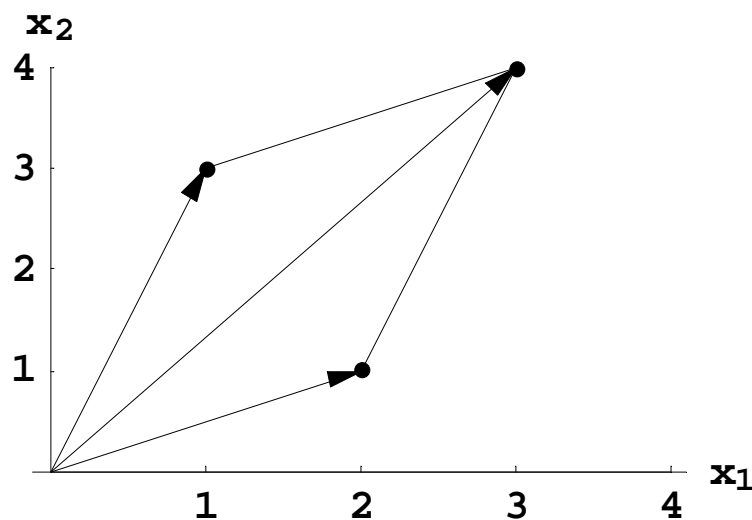
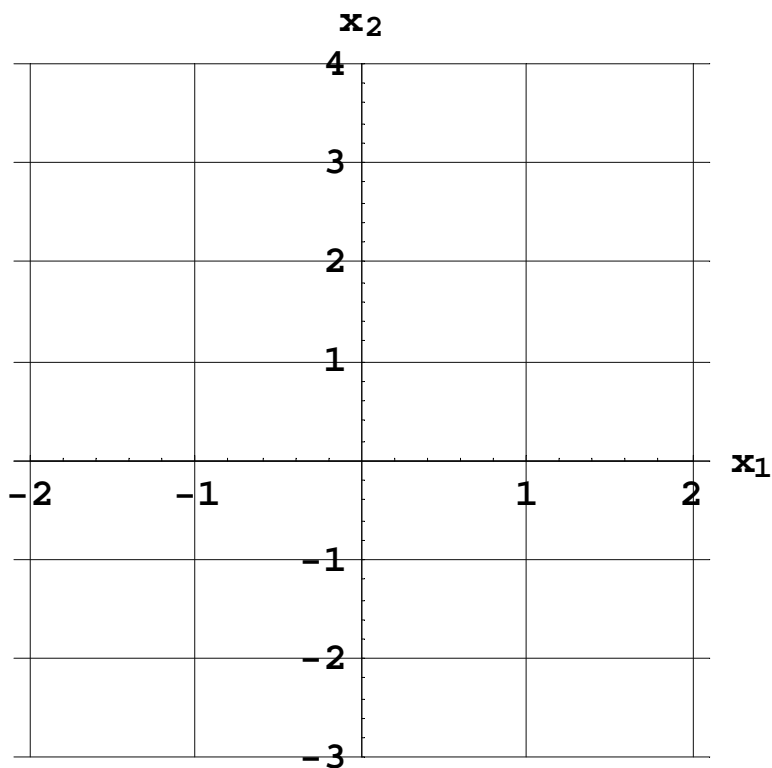


Illustration of the Parallelogram Rule

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express \mathbf{u} , $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbf{R}^n and given scalars c_1, c_2, \dots, c_p , the vector \mathbf{y} defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p$$

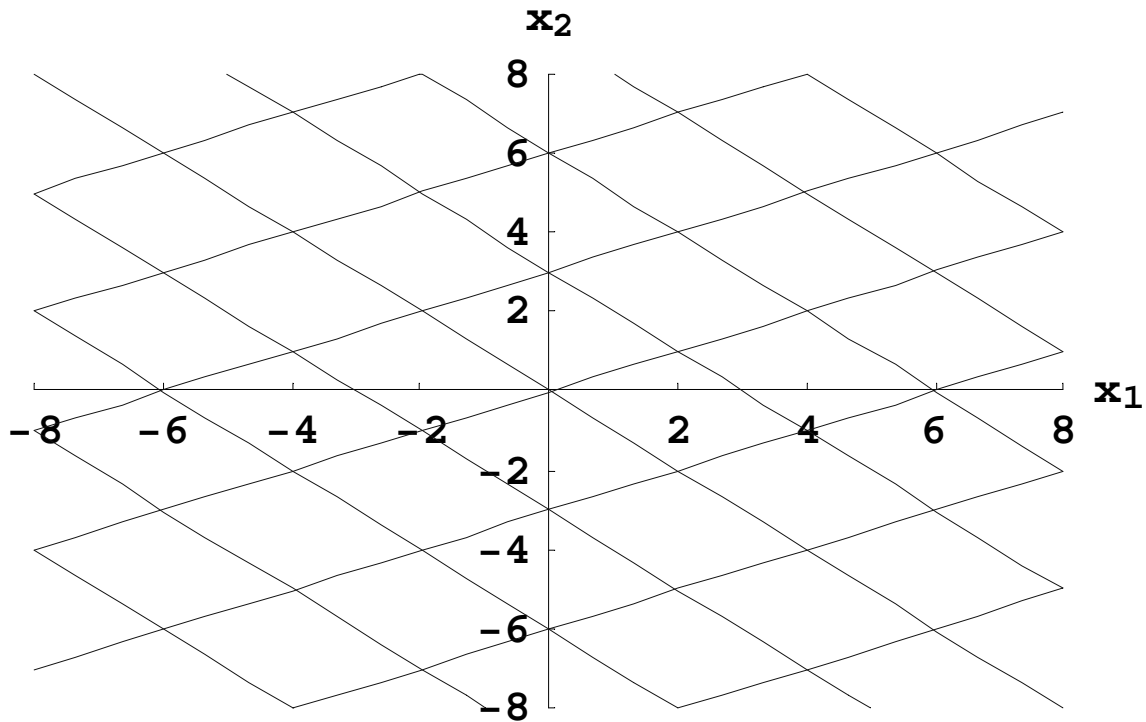
is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ using weights c_1, c_2, \dots, c_p .

Examples of linear combinations of \mathbf{v}_1 and \mathbf{v}_2 :

$$3\mathbf{v}_1 + 2\mathbf{v}_2, \quad \frac{1}{3}\mathbf{v}_1, \quad \mathbf{v}_1 - 2\mathbf{v}_2, \quad \mathbf{0}$$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of \mathbf{v}_1 and \mathbf{v}_2 :

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if \mathbf{b} is a linear combination of \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 .

Solution: Vector \mathbf{b} is a linear combination of \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 if we can find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

Corresponding System:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}$$

Corresponding Augmented Matrix:

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= \underline{\hspace{1cm}} \\ x_2 &= \underline{\hspace{1cm}} \\ x_3 &= \underline{\hspace{1cm}} \end{aligned}$$

Review of the last example: \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 and \mathbf{b} are columns of the augmented matrix

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$
 $\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}.$$

A vector equation

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n = \mathbf{b}$$

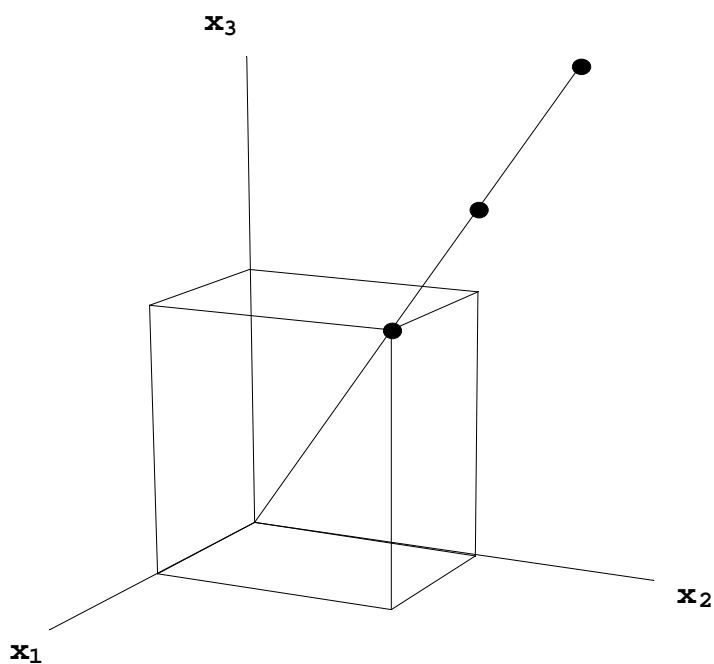
has the same solution set as the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array} \right].$$

In particular, \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

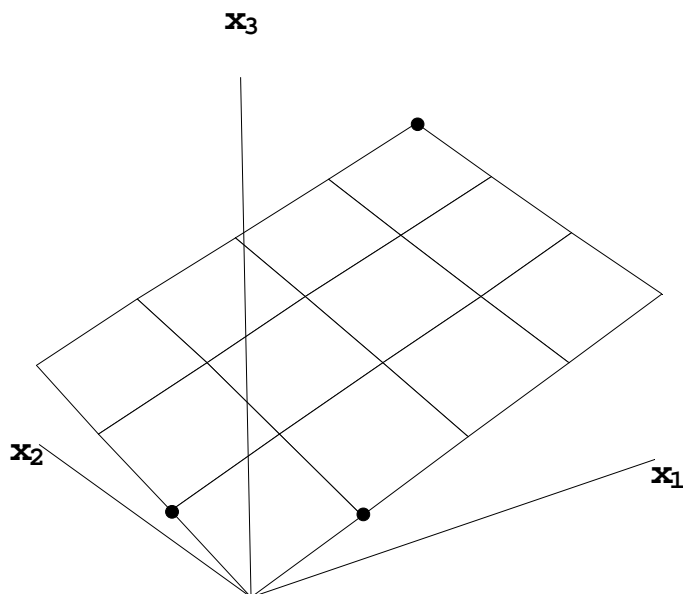
The Span of a Set of Vectors

EXAMPLE: Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with \mathbf{v} , $2\mathbf{v}$ and $1.5\mathbf{v}$ on the graph below.



\mathbf{v} , $2\mathbf{v}$ and $1.5\mathbf{v}$ all lie on the same line.
Span $\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$.
Here, **Span** $\{\mathbf{v}\} =$ a line through the origin.

EXAMPLE: Label \mathbf{u} , \mathbf{v} , $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} + 4\mathbf{v}$ on the graph below.



\mathbf{u} , \mathbf{v} , $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u} + 4\mathbf{v}$ all lie in the same plane.
 $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u} + x_2\mathbf{v}$.
 Here, $\text{Span}\{\mathbf{u}, \mathbf{v}\} =$ a plane through the origin.

Definition

Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ are in \mathbf{R}^n ; then

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\} = \text{set of all linear combinations of } \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p.$$

Stated another way: $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written as

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_p\mathbf{v}_p$$

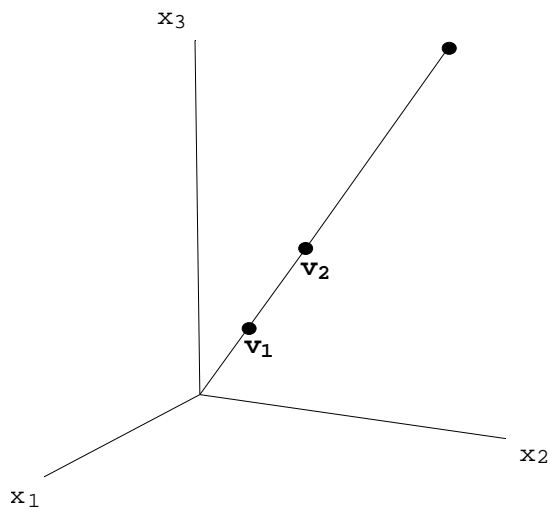
where x_1, x_2, \dots, x_p are scalars.

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find a vector in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

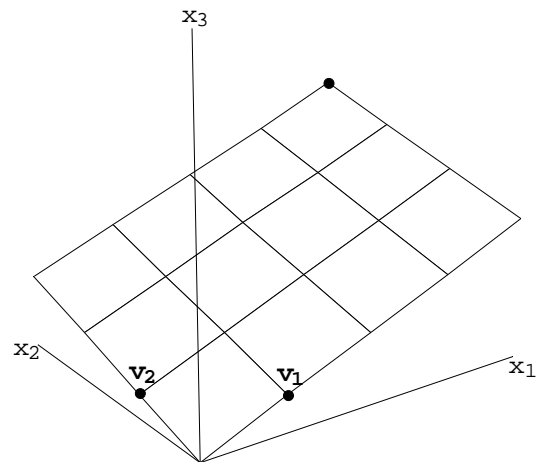
(b) Describe $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically.

Spanning Sets in \mathbb{R}^3



\mathbf{v}_2 is a multiple of \mathbf{v}_1

$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{v}_2\}$
(line through the origin)



\mathbf{v}_2 is **not** a multiple of \mathbf{v}_1

$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{plane through the origin}$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$. Is $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is \mathbf{b} in the plane spanned by the columns of A ?

Solution:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$$

Do x_1 and x_2 exist so that

Corresponding augmented matrix:

$$\begin{bmatrix} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{bmatrix}$$

So \mathbf{b} is not in the plane spanned by the columns of A