

# $K_{1,3}$ -subdivision Tolerance Representations of Cycles

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## Abstract

Consider a simple undirected graph  $G = (V, E)$ . A family of subtrees,  $\{S_v\}_{v \in V}$ , of a tree  $H$  is called an  $(H, t)$ -representation of  $G$  provided  $uv \in E$  if and only if  $|V(S_u) \cap V(S_v)| \geq t$ . In this paper we consider  $(H_m, t)$ -representations for cycles, where  $H_m$  is the  $K_{1,3}$ -subdivision that has exactly one node  $x$  of degree three and exactly three leaves, each of distance  $m$  from  $x$ . We denote the set of  $(H_m, t)$ -representable graphs for some positive integer  $m$ , as  $\mathcal{H}(t)$ . We show that the maximum size of a cycle in  $\mathcal{H}(t)$  is  $3t - 3$  for  $t \in \{3, 4, 5\}$  and is asymptotically equal to  $\frac{1}{4}t^2$  for  $t \geq 6$  and obtain as a corollary that the maximum size of a cycle in the finite triangular lattice with outer boundary of  $t + 1$  vertices on each side is also asymptotically equal to  $\frac{1}{4}t^2$ ,  $t \geq 6$ . We also show  $C_n \in \mathcal{H}(t)$  implies  $C_{n-1} \in \mathcal{H}(t)$  for  $n \geq 9$ .

# 1 Introduction

In this paper we consider tree tolerance representations of graphs.

**Definition 1** Let  $G = (V, E)$  be a graph,  $H$  be a tree and  $t > 0$ . Then  $G$  is called  **$(H, t)$ -representable** if there exists a family of subtrees of  $H$ ,  $\{S_v\}_{v \in V}$ , such that

$$uv \in E \leftrightarrow |V(S_u) \cap V(S_v)| \geq t.$$

In this case we call  $\{S_v\}_{v \in V}$  a *tree tolerance representation of  $G$  with tolerance  $t$*  or an  $(H, t)$ -*representation of  $G$* . Also note that the tree  $H$  is referred to as the **host tree** of the representation and  $t$  is called the **tolerance**.

**Definition 2** The tree  $H_m$  is the  $K_{1,3}$ -subdivision with exactly one node  $x$ , of degree three, and exactly three leaves, each with distance  $m$  from  $x$ , as shown in Figure 1.

We denote the three subpaths of  $H_m$ , beginning at  $x = a_0 = b_0 = c_0$  and ending at each of the leaves, as follows:

$$P_a = a_0, a_1, a_2, \dots, a_m$$

$$P_b = b_0, b_1, b_2, \dots, b_m$$

$$P_c = c_0, c_1, c_2, \dots, c_m$$

We also denote the class of all  $(H_m, t)$ -representable graphs for any  $m$ , as  $\mathcal{H}(t)$ .

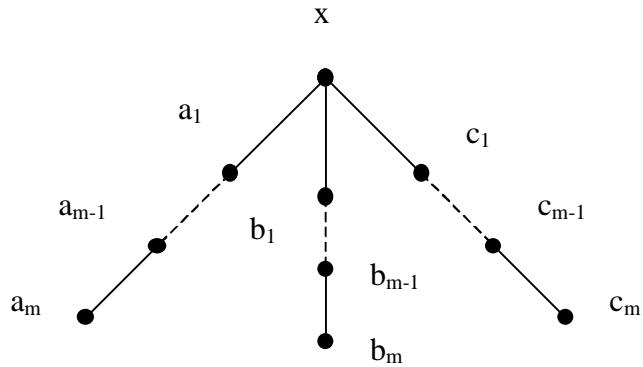


Figure 1:  $H_m$

The question which motivated the work in this paper is: What graphs are in  $\mathcal{H}(t)$ ? We know that when the tolerance is limited to one, the tree representable graphs are the chordal graphs, see [1] [2] and [7]. The set of graphs that are representable by a path has been completely characterized. This is due to the fact that the set of path representable graphs is equivalent to the set of interval graphs, which was characterized by Lekkerkerker and Boland in [5]. In their characterization, Lekkerkerker and Boland showed that any cycle of order at least four is one of the forbidden subgraphs of an interval graph. It is also evident that tolerance has no significance in interval representations, a fact due to Golumbic and Monma in [3] and rephrased in terms of path representations by Jamison and Mulder in [4].

As a next step, it is natural to consider what cycles can be represented using  $H_m$  and tolerance  $t \geq 2$ , since  $H_m$  can be viewed as two paths, where an end-

vertex of one path is identified with an internal vertex of the other. Figure 2 illustrates an  $H_2$  tolerance representation of  $C_4$  with tolerance 3.

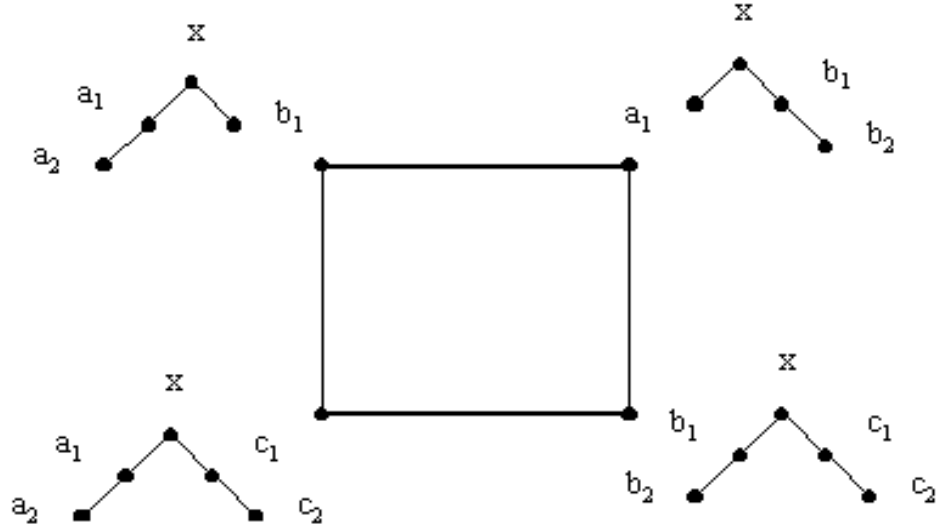


Figure 2:  $(H_2, 3)$ -representation of  $C_4$

## 2 Cycles in $\mathcal{H}(t)$

### 2.1 Properties of $(H_m, t)$ -Representations for Cycles

In this section we will address the necessary preliminaries, including some very useful properties of any  $(H_m, t)$ -representation for cycles of size at least four. First, let us assume that  $V(C_n) = \{1, 2, \dots, n\}$  where consecutive numbers are adjacent and 1 and  $n$  are adjacent. For clarity of exposition, we will refer to the vertices

of the cycles as vertices, and the vertices of the host tree  $H_m$  as nodes.

**Lemma 1** *If  $C_n \in \mathcal{H}(t)$  for  $n \geq 4$ , with  $(H_m, t)$ -representation  $\{S_i\}_{i=1}^n$ , then  $\deg_{S_i}(x) \geq 2$  for  $i = 1, 2, \dots, n$ .*

**Proof:** The proof is by induction on  $n$ .

The base case is for  $n = 4$ . For the sake of contradiction, suppose  $C_4 \in \mathcal{H}(t)$  with  $(H_m, t)$ -representation  $\{S_i\}_{i=1}^4$ , such that  $x \notin S_j$  or  $\deg_{S_j}(x) = 1$  for some  $j \in \{1, 2, 3, 4\}$ . Without loss of generality, we can assume  $j = 1$ . So  $S_1$  is a subpath of  $P_a, P_b$ , or  $P_c$ . Suppose  $S_1 \subseteq P_a$ . Then  $S_2$  and  $S_4$  must each contain at least  $t$  nodes from  $P_a$ . Hence, either  $x \notin S_2$  or  $x \notin S_4$ ; otherwise,  $|S_2 \cap S_4| \geq t$ , which is impossible. Assume  $x \notin S_2$ , which means  $S_2 \subseteq P_a$ .

If  $x \notin S_4$  or  $\deg_{S_4}(x) = 1$ , then  $S_4 \subseteq P_a$ . In this case, if  $S_3$  has any nodes that are not in  $P_a$ , then they are unnecessary and can be removed, leaving us with a path representation of  $C_4$ , which is impossible.

If  $\deg_{S_4}(x) = 2$ , then without loss of generality, we can assume  $S_4 \subseteq P_a \cup P_b$ . If  $S_3$  contains any nodes from  $P_c$  other than  $x$ , then we can remove them as they are unnecessary. This also gives us a path representation of  $C_4$ .

Suppose the  $\deg_{S_4}(x) = 3$ . Then we can assume that any nodes of  $S_4$  that are not in  $P_a$ , are also in  $S_3$ ; otherwise, they are unnecessary and can be removed. So  $S_3$  must contain at least 2 nodes from  $P_a$  in order to intersect with  $S_2$ . Hence,  $\deg_{S_3}(x) = 3$  as well, since  $t \geq 2$ . Remove any nodes of  $P_c$  except  $x$ , from

$V(S_3) \cap V(S_4)$  and add this same number of nodes of  $P_b$  to  $V(S_3) \cap V(S_4)$ . This creates a path representation of  $C_4$ . Once again, we have a contradiction.

Assume  $n \geq 4$  and Lemma 1 holds for  $n - 1$ . For the sake of contradiction, suppose  $C_n \in \mathcal{H}(t)$  with  $(H_m, t)$ -representation  $\{S_i\}_{i=1}^n$ , such that  $x \notin S_j$  or  $\deg_{S_j}(x) = 1$  for some  $j \in \{1, 2, \dots, n\}$ . We can assume  $j = 1$ , and  $S_1 \subseteq P_a$ . Observe that  $S_2$  and  $S_n$  must each contain at least  $t$  nodes from  $P_a$ . Hence, either  $x \notin S_2$  or  $x \notin S_n$ ; otherwise,  $|S_2 \cap S_n| \geq t$ , which is impossible. Assume  $x \notin S_2$ , which means  $S_2 \subseteq P_a$ . Contract the edge between vertex 1 and 2 in  $C_n$  and denote the new vertex as  $w$ . Now we have a new cycle of size  $n - 1$  for which we create an  $(H_m, t)$ -representation as follows: let  $S_w = S_1 \cup S_2$  and leave the other representing subtrees unchanged. Since  $S_1$ ,  $S_2$ , and  $S_w = S_1 \cup S_2$  are paths, we know that if both (1)  $(S_1 \setminus S_2) \cap S_i \neq \emptyset$  and (2)  $(S_2 \setminus S_1) \cap S_i \neq \emptyset$ , then  $S_1 \cap S_2 \subseteq S_i$ . But since  $|S_1 \cap S_2| \geq t$ , we know that (1) and (2) cannot both be true. Therefore, if  $|S_i \cap S_w| \geq t$  then either  $|S_i \cap S_1| \geq t$  or  $|S_i \cap S_2| \geq t$ . Hence we have an  $(H_m, t)$ -representation for  $C_{n-1}$  containing a representing subtree,  $S_w$ , with  $x$  of degree one, which contradicts our induction assumption.  $\square$

We know  $C_n \notin \mathcal{H}(1)$  for  $n \geq 4$ . Also, from Lemma 1 it is easy to see that  $C_n \notin \mathcal{H}(2)$  for  $n \geq 4$ . However, when we raise the tolerance above two, we are able to represent certain cycles, as we show in section 2.2.

**Lemma 2** *Let  $n$  be the maximum value such that  $C_n \in \mathcal{H}(t)$ . Then  $C_n$  has an*

$(H_m, t)$ -representation,  $\{S_i\}_{i=1}^n$ , such that the following conditions hold:

(i)  $|V(S_i) \cap V(S_{i+1})| = t$ , for all  $i \in \{1, 2, \dots, n-1\}$ ,  $|V(S_n) \cap V(S_1)| = t$ , and

(ii)  $|V(S_i)| = t + 1$ , for all  $i \in \{1, 2, \dots, n\}$ .

**Proof:** Let  $n$  be the maximum value such that  $C_n \in \mathcal{H}(t)$ . Let  $\{S_i\}_{i=1}^n$  be an  $(H_m, t)$ -representation for  $C_n$ . To prove (i), we suppose  $|V(S_1) \cap V(S_2)| \geq t + 1$  and show that we can reduce  $|V(S_1) \cap V(S_2)|$  by one and still have an  $(H_m, t)$ -representation for  $C_n$ .

First, remove the nodes of  $S_2$  that are not in  $V(S_1)$  or  $V(S_3)$ . Observe that the removal of these nodes does not disconnect  $S_2$ . Indeed, from Lemma 1, we know that  $x$  is in every representing subtree. Also, for any  $i \in \{1, 2, \dots\}$ , if for some  $v \in V(C_n)$ ,  $a_i \notin S_v$  ( $b_i \notin S_v$ ,  $c_i \notin S_v$ ) then for all  $j > i$  we have  $a_j \notin S_v$  ( $b_j \notin S_v$ ,  $c_j \notin S_v$ , respectively). Hence, any nodes we may have removed must form disjoint subpaths of  $S_2$ , each containing a leaf of  $S_2$  and not containing  $x$ .

Observe that there are at least two nodes in  $V(S_1) \cap V(S_2)$  that are not in  $V(S_3)$ . Indeed, as if not, then  $|V(S_1) \cap V(S_3)| \geq t$ , since  $|V(S_1) \cap V(S_2)| \geq t + 1$ . Without loss of generality, suppose one of these nodes is in  $P_a$  and denote it  $a_i$ . Hence,  $a_i \in [V(S_1) \cap V(S_2)] / V(S_3)$ . Now since  $x$  is in all the representing subtrees, we know  $a_j \notin V(S_3)$  for  $j \geq i$ . This means that there is a leaf of  $S_1 \cap S_2$ ,  $a_k$ , in  $[V(S_1) \cap V(S_2)] / V(S_3)$ . Observe that  $a_k$  is also a leaf in  $S_2$ , since we removed the

nodes of  $S_2$  that were not in  $V(S_1)$  or  $V(S_3)$ . Now remove  $a_k$  from  $S_2$ , thereby reducing  $|V(S_1) \cap V(S_2)|$  by one, and we still have an  $(H_m, t)$ -representation for  $C_n$ , as was to be shown.

To prove (ii) we suppose  $|V(S_1)| \geq t + 2$  and show that we can replace  $S_1$  with two subtrees, yielding a representation for a cycle of size  $n + 1$ . First, remove the nodes of  $S_1$  that are not in  $S_n$  or  $S_2$ , as they are unnecessary. Once again, note that this process does not disconnect  $S_1$  by the same reasoning as used in the proof of (i). From (i) we have  $|V(S_1) \cap V(S_n)| = |V(S_1) \cap V(S_2)| = t$ . There exist two vertices,  $l_1$  and  $l_2$ , such that  $l_1$  is a leaf of  $S_1 \cap S_2$  in  $[V(S_1) \cap V(S_2)]/V(S_n)$  and  $l_2$  is a leaf of  $S_1 \cap S_2$  in  $[V(S_1) \cap V(S_n)]/V(S_2)$ . Now replace  $S_1$  with  $S_1/\{l_1\}$  and  $S_1/\{l_2\}$ , which intersect each other in at least  $t$  nodes, since  $|V(S_1)| \geq t + 2$ . Hence, we have an  $(H_m, t)$ -representation for a larger cycle than  $C_n$ .  $\square$

**Proposition 1** *Let  $n$  be the maximum value such that  $C_n \in \mathcal{H}(t)$ . If  $C_n$  has an  $(H_m, t)$ -representation for some value  $m$ , then  $C_n$  has an  $(H_{t-1}, t)$ -representation.*

Proposition 1 follows from Lemma 1 and Lemma 2 part (ii).

## 2.2 Main Results

We start by addressing some notation before our main theorem. Consider a subtree  $S_v$  of  $H_{t-1}$  for some  $t$ . In subsequent results, we will denote  $S_v$  by  $(i, j, k)$  where  $i, j$ , and  $k$  are the maximum values, such that  $a_i, b_j$ , and  $c_k \in V(S_v)$ . Now

consider the graph  $G_t = (V_t, E_t)$  where the vertex set  $V_t$  is the set of all subtrees  $(i, j, k)$  of  $H_{t-1}$  such that  $i + j + k = t$  and at most one of  $\{i, j, k\}$  is zero. The edge set  $E_t$  consists of the pairs of vertices  $S_u = (i_u, j_u, k_u)$  and  $S_v = (i_v, j_v, k_v)$  such that

$$\min\{i_u, i_v\} + \min\{j_u, j_v\} + \min\{k_u, k_v\} = t - 1. \quad (1)$$

Note that this condition is equivalent to  $uv \in E \leftrightarrow |V(S_u) \cap V(S_v)| \geq t$ .

By Lemma 1, Lemma 2 and Proposition 1, the maximum size of a cycle in  $\mathcal{H}(t)$  is equal to the maximum size of an induced cycle in  $G_t$ .

**Definition 3** *We denote by  $T_t$  the finite sublattice, of the infinite triangular lattice, with an equilateral triangle as the outer boundary of  $t + 1$  vertices on each side.*

Note that  $G_t$  is  $T_t$  with the three corners removed. We obtain a labeling for the vertices of  $T_t$  by considering the points having integer coordinates in the plane  $x + y + z = t$ , restricted to  $x, y, z \geq 0$ . These points correspond to the vertices of the lattice. The edges of the lattice are between 2 points whose distance apart in the plane is 1. See Figure 3 for an illustration of  $G_t$ . Note that row  $i$  contains the vertices that have  $i$  fixed in the second coordinate. Theorem 1 addresses tolerances  $t = 3, 4$ , and 5 in which cases the induced subgraph of all vertices with zero in exactly one coordinate is the maximum induced cycle in  $G_t$ .

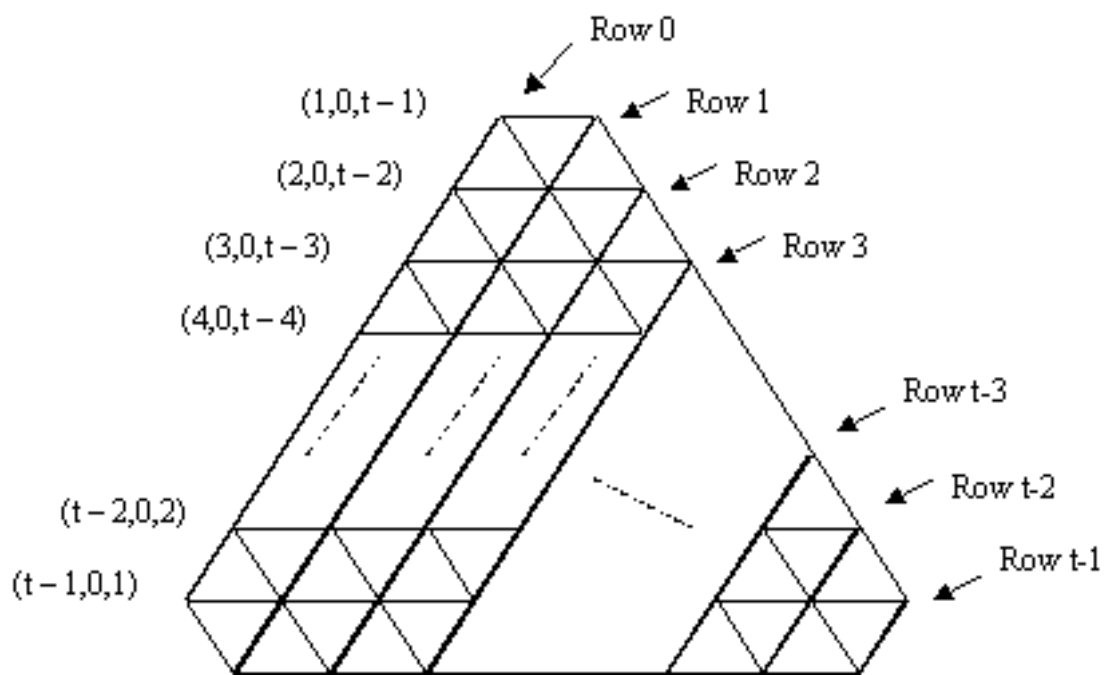


Figure 3:  $G_t$

**Theorem 1** For tolerance  $t = 3, 4$ , and  $5$ , the maximum value of  $n$  for which  $C_n \in \mathcal{H}(t)$  is  $3t - 3$ .

In lieu of a proof, we provide a diagram of  $C_{12}$  in  $G_5$ . See Figure 4.

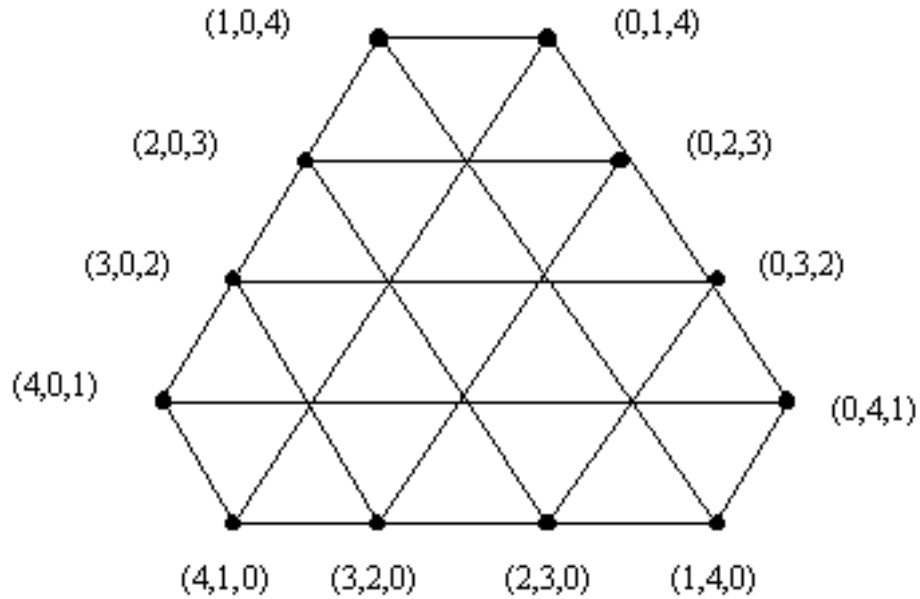


Figure 4: Largest cycle in  $G_5$

When  $t = 6$  there is an induced cycle of size at least 16 in  $G_6$ , which is larger than  $3t - 3 = 15$ . Such a cycle is obtained by including vertices  $(3, 2, 1)$  and  $(2, 3, 1)$  as shown in Figure 5. As we increase the tolerance  $t$  even more, we can use additional interior vertices of the corresponding triangular grid  $G_t$  to come up with larger cycles.

We have the following theorem.

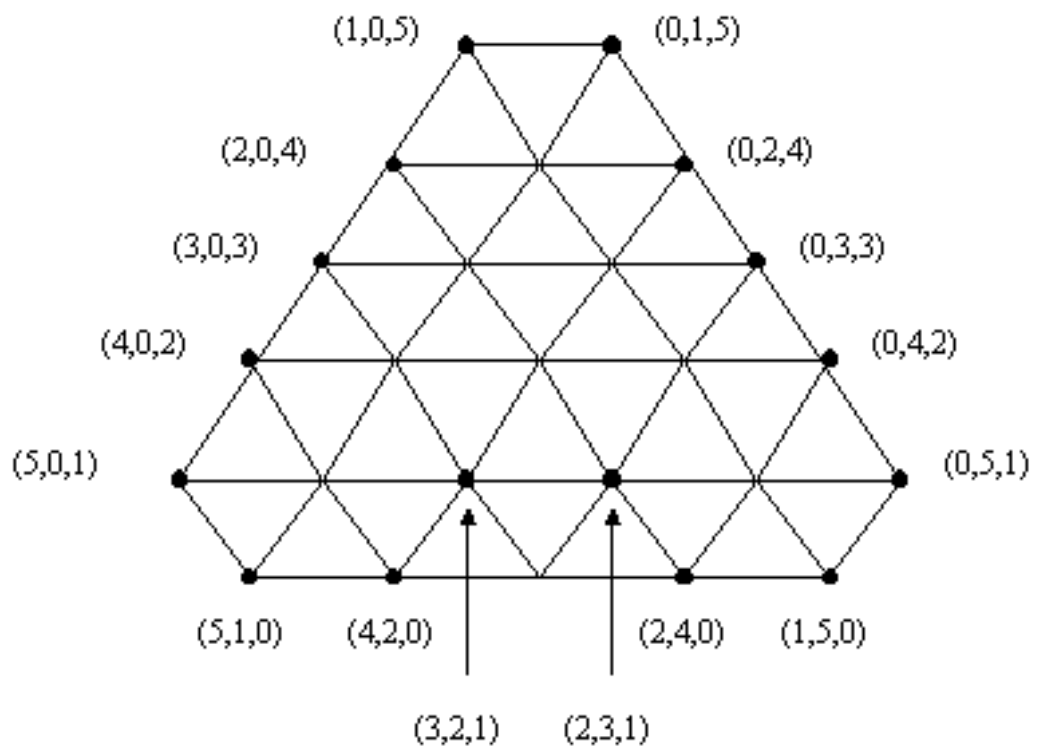


Figure 5:  $C_{16}$  in  $G_6$

**Theorem 2** *Let  $n$  be the maximum value for which  $C_n \in \mathcal{H}(t)$  and let  $t \geq 6$ .*

*Then*

$$\frac{1}{4}t^2 + t + \frac{3}{4} \leq n \leq \frac{1}{4}t^2 + \frac{3}{2}t - \frac{3}{4}.$$

**Proof:** We will refer to the row labeling shown in Figure 3.

Let  $n$  be the maximum size of an induced cycle in  $G_t$ . First we show  $n \geq \frac{1}{4}t^2 + t + \frac{3}{4}$  by constructing a cycle of size  $\frac{1}{4}t^2 + t + \frac{3}{4}$  in  $G_t$ . The general idea behind the construction is to include as many vertices as possible from every other row, starting with Row 0 and we use two vertices from each of the remaining rows to complete the cycle. Thus the cycle contains all the vertices of Row 0,  $(1, 0, t-1), (2, 0, t-2), \dots, (t-1, 0, 1)$ , vertices  $(t-1, 1, 0)$  and  $(0, 1, t-1)$  from Row 1, all vertices except  $(1, 2, t-3)$  and  $(2, 2, t-4)$  from Row 2, the two vertices  $(0, 3, t-3)$  and  $(2, 3, t-5)$  from Row 3, and so on.

As we follow the cycle down Row 0, up Row 2, down Row 4, up Row 6, we see that the last up row depends on the value of  $t \pmod{4}$ .

When  $t = 3 \pmod{4}$ , the last up row is Row  $t-1$ . Counting the vertices by grouping two rows together, we get that the number of vertices used is

$$(t+1) + (t-1) + (t-3) + \dots + 4 + 2 = \frac{1}{4}t^2 + t + \frac{3}{4}.$$

When  $t = 0 \pmod{4}$ , the last up row is Row  $t-2$ . Counting the vertices by

grouping two rows together, we get that the number of vertices used is

$$(t + 1) + (t - 1) + (t - 3) + \cdots + 5 + 3 = \frac{1}{4}t^2 + t.$$

In this case we can elongate the cycle by 1 vertex by removing vertex  $(1, t - 2, 1)$  and including  $(1, t - 1, 0)$  and  $(0, t - 1, 1)$ , obtaining  $n \geq \frac{1}{4}t^2 + t + 1$ .

When  $t \equiv 1 \pmod{4}$ , the last up row is Row  $t - 3$ . Counting the vertices by grouping two rows together, we get that the number of vertices used is

$$(t + 1) + (t - 1) + (t - 3) + \cdots + 6 + 4 = \frac{1}{4}t^2 + t - \frac{5}{4}.$$

In this case we can elongate the cycle by 2 vertices by removing vertices  $(2, t - 3, 1)$  and  $(1, t - 3, 2)$  and including  $(0, t - 2, 2)$ ,  $(2, t - 2, 0)$ ,  $(1, t - 1, 0)$  and  $(0, t - 1, 1)$ , obtaining  $n \geq \frac{1}{4}t^2 + t + \frac{3}{4}$ .

When  $t \equiv 2 \pmod{4}$ , the last up row is Row  $t - 4$ . Counting the vertices by grouping two rows together, we get that the number of vertices used is

$$(t + 1) + (t - 1) + (t - 3) + \cdots + 7 + 5 = \frac{1}{4}t^2 + t - 3.$$

In this case we can elongate the cycle by 4 vertices by removing vertices  $(2, t - 4, 2)$  and  $(1, t - 4, 3)$  and including  $(0, t - 3, 3)$ ,  $(2, t - 3, 1)$ ,  $(0, t - 2, 2)$ ,  $(2, t - 2, 0)$ ,  $(1, t - 1, 0)$  and  $(0, t - 1, 1)$ , obtaining  $n \geq \frac{1}{4}t^2 + t + 1$ .

Now we show  $n \leq \frac{1}{4}t^2 + \frac{3}{2}t - \frac{3}{4}$ . We look at the embedding of  $G_t$  in  $G_{t+3}$  where each vertex of  $G_t$  has degree 6 in  $G_{t+3}$ . Assume  $C_n$  is an induced cycle of

$G_t$ . Let  $\mathcal{A}$  be the set of triangular faces in  $G_{t+3}$ , so that  $|\mathcal{A}| = t^2 + 6t + 6$ . Let  $T = \sum_{A \in \mathcal{A}} |V(C_n) \cap V(A)|$ .

Each vertex of  $C_n$  is on 6 triangles in  $\mathcal{A}$ . Each triangular face contains at most 2 vertices from  $C_n$ , but for each vertex used in  $C_n$ , there must be 2 members of  $\mathcal{A}$  that have only this one vertex on them. Also, the 9 triangular faces from the corners of  $G_{t+3}$  will not have any cycle vertices on them. Summing over all triangular faces, we see that  $T \leq 2(t^2 + 6t - 3) - 2n$ . When we sum over the vertices of  $C_n$ , we see that  $6n = T$  and we have  $8n \leq 2t^2 + 12t - 6$ . Thus,  $n \leq \frac{1}{4}t^2 + \frac{3}{2}t - \frac{3}{4}$ .

□

**Corollary 1** *Let  $t \geq 6$  and  $n$  be the maximum size of an induced cycle in  $T_t$ . Then  $n$  is asymptotically equal to  $\frac{1}{4}t^2$ .*

### 3 Short Cycles

**Theorem 3** *If  $C_n \in \mathcal{H}(t)$  then  $C_n \in \mathcal{H}(t + 1)$ .*

Theorem 3 is clear because  $G_t$  is an induced subgraph of  $G_{t+1}$ .

Considering cycles  $C_n$  where  $n \geq 4$ , we know that  $C_4, C_5$ , and  $C_7$  are not induced cycles of  $G_t$  for any  $t$  and  $G_1$  and  $G_2$  have none. The only induced cycle in  $G_3$  is  $C_6$  and  $G_4$  contains only  $C_6, C_8$ , and  $C_9$  as induced cycles. For the general

result, we need the following definition.

**Definition 4** A **2-point turn** of  $G_t$  is a path composed of three subpaths described as follows: All vertices of one subpath contain a fixed 1st coordinate, all vertices of another subpath contain a fixed second coordinate and all vertices of the remaining subpath contain a fixed third coordinate. Let the middle subpath have coordinate  $i$  fixed at the value  $b$ . The two end subpaths will each contain vertices with coordinate  $i$  having the value of  $b$ . The other vertices in the two end subpaths will either contain values of coordinate  $i$  increasing, that is  $b, b + 1, \dots$ , etc., or decreasing,  $b, b - 1, \dots$ .

Consider an induced  $C_n$  in  $G_t$  that has a 2-point turn. We can reduce  $C_n$  by one vertex as follows: Delete the vertices of the middle subpath, which transforms the 2-point turn into two disjoint paths in  $G_t$  and say coordinate  $i$  has the value  $b$ . Denote the two vertices that were adjacent to the middle subpath of the 2-point turn as  $u$  and  $v$ . Then add the vertices on the shortest path between  $u$  and  $v$ . This path will either have coordinate  $i$  equal to  $b + 1$  or  $b - 1$  whatever the case may be. If the new cycle  $C_{n-1}$  is also an induced cycle, we call the 2-point turn so described, a *reducible 2-point turn*. See Figure 6 for an illustration of such a reduction.

In the proof of the following result, when possible, we will use a reducible 2-point turn to reduce the cycle by one vertex.

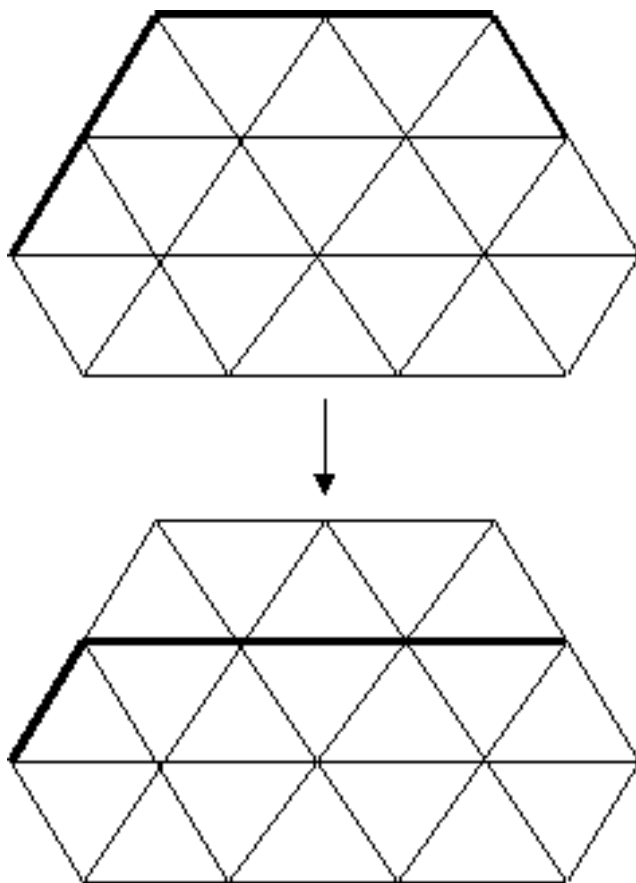


Figure 6: Reducing  $C_n$  by one vertex

**Theorem 4** *If  $n \geq 9$  and  $C_n \in \mathcal{H}(t)$  then  $C_{n-1} \in \mathcal{H}(t)$ .*

**Proof:**

The proof is by induction on  $t$ . For the base case we notice that  $C_8$  and  $C_9$  are the only induced cycles of  $G_4$ .

Let  $t > 4$ ,  $n \geq 9$ , and assume  $C_n$  is an induced subgraph of  $G_t$ . Let us call the set of edges of  $G_t$  that have 0 in the first coordinate of both vertices,  $S(1)$ , the set of edges of  $G_t$  that have 0 in the second coordinate of both vertices,  $S(2)$ , and the set of edges of  $G_t$  that have 0 in the third coordinate of both vertices,  $S(3)$ . We call edges  $\{(1, 0, t), (0, 1, t)\}$ ,  $\{(0, t, 1), (1, t, 0)\}$ , and  $\{(t, 0, 1), (t, 1, 0)\}$  the corner edges.

We know that edges from each of  $S(1)$ ,  $S(2)$ , and  $S(3)$  are used in  $C_n$  or else we would have  $C_n \in \mathcal{H}(t-1)$ , then by induction we would have  $C_{n-1} \in \mathcal{H}(t-1)$ , and then by Theorem 3,  $C_{n-1} \in \mathcal{H}(t)$ .

Let  $H$  be the subgraph of  $G_t$  induced on  $V(G_t) \setminus V(C_n)$ . Viewing  $G_t$  as a planar graph, the interior of  $C_n$  provides a component of  $H$ , which we call the *interior component*. We also know that there are other components of  $H$  that are on the exterior of  $C_n$ , we call them *exterior components*.

If the interior component is a single point or a single edge then  $n = 6$  or  $n = 8$  respectively, but we know  $n \geq 9$ . If an exterior component is a single edge or a path,  $C_n$  must have a reducible 2-point turn and hence we can reduce the cycle

by 1. If any component of  $H$  is any type of single non-edge block, there must be a reducible 2-point turn in  $C_n$  and hence we can reduce the cycle by 1.

We now assume each component has at least two end-blocks in the block-tree diagram and the exterior components are not paths.

Consider the interior component. If there is an end-block that is not a single edge we find a reducible 2-point turn in  $C_n$  and so we can reduce the cycle by 1.

Now, we assume all end-blocks in the interior component are single edges and there are at least two of them,  $e_1, e_2$ . Observe that we can reduce the cycle by 2 by removing one of these edges, say  $e_1$ , from the interior component and shrinking the cycle around the new interior component. The cycle will use the end-vertex of the edge end-block.

Our strategy to complete the proof will be to increase the cycle by one vertex at one edge end-block and decrease by two vertices at another or in some cases, decrease by one at one of the end-blocks.

Starting with one corner edge,  $\{(1, 0, t), (0, 1, t)\}$ , we see that  $C_n$  must use this edge or if not, some edge parallel to this one. Among all edges in  $C_n$  parallel to this one, choose  $\{(a, 0, c), (a - 1, 1, c)\}$  such that  $c$  is largest. We know such an edge exists in  $C_n$  since  $C_n$  contains vertices from  $S(2)$ . Then because the end-blocks of the interior component of  $H$  are edges,  $C_n$  must use  $(a + 1, 0, c - 1)$  and  $(a - 1, 2, c - 1)$  and either  $(a + 2, 0, c - 2)$  and  $(a, 2, c - 2)$  or  $(a + 1, 1, c - 2)$  and

$(a - 1, 3, c - 2)$ . We consider expanding the end-block into a triangle and adjust the cycle to go around the triangle by replacing the vertex used from  $(a, 2, c - 2)$  and  $(a + 1, 1, c - 2)$  with the one not used from  $(a + 2, 0, c - 2)$  and  $(a - 1, 3, c - 2)$  and including either  $(a + 2, 1, c - 3)$  or  $(a, 3, c - 3)$ , whichever one is adjacent to the other vertex just added. If this gives us an induced cycle, either it increased by 1 vertex or remained the same length. If it increased by 1, we decrease the cycle at the other end-block by 2 vertices and we are done. If not, we now decrease by 1 since our end-block is a triangle.

If the above process does not result in an induced cycle, then it must be that one of the two new vertices is adjacent to another part of the cycle.

**case 1** *Vertex  $(a + 2, 0, c - 2)$  is the one being added.*

The vertex  $(a + 2, 0, c - 2)$  cannot be adjacent to another part of the cycle because of its location on the edge. So the other new vertex must be causing the problem and in particular  $(a + 2, 1, c - 3)$  is not on the original cycle. It however is adjacent to another part of the cycle. We know it is adjacent to  $(a + 1, 2, c - 3)$  which is on the original cycle, but this chord does not cause a problem.

That means one or both of  $(a + 2, 2, c - 4)$  and  $(a + 3, 1, c - 4)$  are on the cycle and are creating chords with  $(a + 2, 1, c - 3)$ . Let's take the 3 cases. If just  $(a + 2, 2, c - 4)$  is on the original cycle then remove  $(a + 1, 2, c - 3)$  and we have a new cycle the same length as original. But the end-block is not a single edge so

we have a reducible 2-point turn. If both  $(a + 2, 2, c - 4)$  and  $(a + 3, 1, c - 4)$  are on the original cycle then the exterior component of the original cycle contains  $(a + 2, 0, c - 2)$ ,  $(a + 2, 1, c - 3)$ , and  $(a + 3, 0, c - 3)$  so there would have been a reducible 2-point turn on the original cycle. Now consider just  $(a + 3, 1, c - 4)$  on the original cycle. Then  $(a + 4, 0, c - 4)$  must also be on the cycle. If this is the case, we choose the edge end-block from  $H$  that has the end-vertex  $(a + 4, 1, c - 5)$  to reduce the cycle by 2, making the process of the addition of 1 vertex and deletion of 2 vertices result in an induced cycle of length  $n - 1$ .

**case 2** *Vertex  $(a - 1, 3, c - 2)$  is the one being added.*

Then vertex  $(a, 3, c - 3)$  is also added which might already be on  $C_n$ . Among all edges on  $C_n$  parallel to  $\{(1, 0, t - 1), (0, 1, t - 1)\}$ , choose  $\{(1, b', c'), (0, b' + 1, c')\}$  such that  $c'$  is largest. If  $a = 1$ , then we choose the same edge as before,  $(a, 0, c) = (1, b', c')$ , and  $c = t - 1$ . In this case, we have the same situation as before only in reverse. If  $2 \leq a \leq 4$ , then  $b' \geq 4$ . We first increase by 2 at the end-block that has end-vertex  $(a, 1, c - 1)$  and decrease by 2 at the end-block that has end-vertex  $(1, b' + 1, c' - 1)$ . We notice that the increased portion of the cycle could not have “bumped into” another part of the cycle. Then we have a new cycle of length  $n$ . The process is repeated and this time the value of  $a$  will be one less. Eventually, it will equal 1 and the process is terminated.

If  $a \geq 5$  we locate the edge parallel to  $\{(1, 0, t - 1), (0, 1, t - 1)\}$  that has the largest third coordinate. This could be  $\{(a, 0, c), (a - 1, 1, c)\}$ ,  $\{(1, b', c'), (0, b' + 1, c')\}$  or some other edge that doesn't include a vertex in  $S(1)$  or  $S(2)$ . We call this new edge  $\{(a'', b'', c''), (a'' - 1, b'' + 1, c'')\}$ . The vertex  $(a'', b'' + 1, c'' - 1)$  is the end vertex of the an edge end-block of  $H$ .

We increase the cycle by 2 at this end-block and decrease by 2 at the end-block that has end-vertex  $(a, 1, c - 1)$  or  $(1, b' + 1, c' - 1)$  depending on  $(a'', b'' + 1, c'' - 1)$ . Keep repeating this process until we end up with  $\{(1, 0, t - 1), (0, 1, t - 1)\}$  on the cycle, which terminates the process once again.  $\square$

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