Plus is a binary operation on the real numbers. By associativity we know that any grouping of the sum of 3 numbers gives the same result. Prove that for \( n \geq 3 \), any grouping of the \( n \) numbers \( x_1, x_2, \ldots, x_n \) gives the same result. We will prove that every grouping is the same as the natural grouping,

\[
( ( ( x_1 + x_2 ) + x_3 ) + x_4 ) + \cdots + x_{n-1} ) + x_n
\]

where the plus signs are used in order from left to right. So we may write \( \sum_{i=1}^{n} x_i \) and group the terms by parenthesis any way we like.

For \( n = 4 \) here are all of the possible groupings.

\[
\begin{align*}
(x_1 + x_2) + (x_3 + x_4) \\
(x_1 + (x_2 + x_3)) + x_4 \\
((x_1 + x_2) + x_3) + x_4 \\
x_1 + (x_2 + (x_3 + x_4)) \\
x_1 + ((x_2 + x_3) + x_4)
\end{align*}
\]

**Proof:**

The base case is for \( n = 3 \) which is true by associativity.

We use strong induction here. Suppose that for \( k \), where \( 3 \leq k < n \), any grouping of \( k \) numbers gives the same result, namely they are all equal to

\[
( ( ( x_1 + x_2 ) + x_3 ) + x_4 ) + \cdots + x_{k-1} ) + x_k
\]

Thus we can call the sum \( \sum_{i=1}^{k} x_i \) and group the terms any way we like.

Now assume \( n > 3 \) We take an arbitrary grouping of \( x_1, x_2, \ldots, x_n \).

In any grouping, we perform the addition of two numbers at a time. So we can think of the plus signs as being performed in a certain order.

Take the one that is to be performed last. On either side of that plus sign, there are two groupings. One is of \( x_1, x_2, \ldots, x_j \) and the other is of \( x_{j+1}, \ldots, x_n \).

The number of terms in each grouping is less than \( n \). If the number of terms is less than 3, we know that there is no order of operations to worry about. If the number of terms is at least 3, we use the strong induction assumption to conclude that both sides of the the last plus sign can be represented by a sum, with no parenthesis indicated.

We have

\[
\left( \sum_{i=1}^{j} x_i \right) + \left( \sum_{i=j+1}^{n} x_i \right).
\]
We can group together the sums anyway we want. So for the second sum we write: \[ \sum_{i=j+1}^n x_i = x_{j+1} + \left( \sum_{i=j+2}^n x_i \right), \] and for the first sum we assume it is grouped in the natural order.

\[
\begin{align*}
\left( \sum_{i=1}^j x_i \right) + \left( \sum_{i=j+1}^n x_i \right) &= \left( \sum_{i=1}^j x_i \right) + \left( x_{j+1} + \sum_{i=j+2}^n x_i \right) \\
&= \left( \sum_{i=1}^j x_i + x_{j+1} \right) + \sum_{i=j+2}^n x_i \\
&= \sum_{i=1}^{j+1} x_i + \left( \sum_{i=j+2}^n x_i \right)
\end{align*}
\]

We continue in this way breaking off the first term of the second sum and grouping it together with the first sum until there is one term left and the natural grouping is assumed on the first sum, that is the order of performing the additions is from left to right. At that point, addition is used to obtain a sum of the terms in the natural order. Here is the last step:

\[
\begin{align*}
\left( \sum_{i=1}^{n-1} x_i \right) + \left( \sum_{i=n}^n x_i \right) &= \left( \sum_{i=1}^{n-1} x_i \right) + x_n \\
&= \sum_{i=1}^n x_i
\end{align*}
\]