

Brendan Alarie

Section 1.2: #8: For any two finite sets  $S$  and  $T$ , show that the average of  $|S|$  and  $|T|$  does not exceed  $|S \cup T|$ .

The average of  $|S|$  and  $|T|$  is  $\frac{|S| + |T|}{2}$ .  
Since  $S \cup T$  contains all of the elements in  $S$  and all of the elements in  $T$ ,  $S \subseteq S \cup T$  and  $T \subseteq S \cup T$ . Thus by theorem 1.2.1 part c,  $|S| \leq |S \cup T|$  and  $|T| \leq |S \cup T|$ .

So:  $|S| + |T| \leq |S \cup T| + |S \cup T|$ .

$$|S| + |T| \leq 2|S \cup T|$$

$$\frac{|S| + |T|}{2} \leq |S \cup T|$$

Therefore, the average of  $|S|$  and  $|T|$  does not exceed  $|S \cup T|$ .

good!

Amy Brown

**Proposition 1(3):**

$$(-x)y = -(xy)$$

In order to prove Proposition 1(3) we need to show that  $(-x)y$  is the additive inverse of  $xy$ . To show this we need to prove that  $(-x)y + (xy) = 0$ . Using the distributive law we have that  $(x + (-x))y = 0$ . Using the additive inverse we get  $0 \cdot y = 0$ . By the commutativity property we can say that  $y \cdot 0 = 0$ . Using Proposition 1(2), which states that  $y \cdot 0 = 0$ , we have that  $0 = 0$ . This tells us that  $(-x)y$  is the additive inverse of  $xy$ . The additive inverse of  $xy$  is  $-(xy)$ . This proves that  $(-x)y = -(xy)$



Diejun Xu

**Section 1-4 #9** Use a truth table to verify that  $(P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (\sim P \vee Q)$  is a tautology.

**Proof:**

| P | Q | $\sim P$ | $\sim Q$ | $P \wedge \sim Q$ | $\sim P \wedge Q$ | $\sim P \vee Q$ | $(P \wedge \sim Q) \vee (\sim P \wedge Q)$ | $(P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (\sim P \vee Q)$ |
|---|---|----------|----------|-------------------|-------------------|-----------------|--|---|
| T | T | F        | F        | F                 | F                 | T               | F  | T   |
| T | F | F        | T        | T                 | F                 | F               | T  | T   |
| F | T | T        | F        | F                 | T                 | T               | T  | T   |
| F | F | T        | T        | F                 | F                 | T               | F  | T   |

Since the statement is always true, so it is a tautology.

Andrea Medeiros

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**Given  $2 \leq y + z$ , show that  $4x + 4yz \leq (x + y + z)^2$ . Also assume  $x \geq 0$ .**

Theorem 1 of the AGM Inequality states;

"If  $x, y$  are an element of the real numbers, then  $xy \leq \frac{(x+y)^2}{(2)^2}$ ".

If we let  $x = y$  and  $y = z$  we get the following;

$$yz \leq \frac{(y+z)^2}{(2)^2}$$

$$yz \leq \frac{(y+z)^2}{4}$$

$$4yz \leq (y+z)^2$$

Also  $(x+(y+z))^2 = x^2 + 2x(y+z) + (y+z)^2$ . We can add  $x^2 + 2x(y+z)$  to both sides of this inequality to get the following;

$$x^2 + 2x(y+z) + 4yz \leq (x+y+z)^2$$

Now we are going to use what we are given which is,  $2 \leq y + z$ . Since  $x \geq 0$  we can multiply  $2x$  to both sides of this inequality. As a result we get the following inequality:

$$2(2x) \leq 2x(y+z).$$

$$4x \leq 2x(y+z).$$

So now we have  $x^2 + 4x + 4yz \leq (x+y+z)^2$ . We also can use Proposition 4(4), which states,  $0 \leq x^2$  which is true for all  $x$ . So we can say  $4x + 4yz \leq (x+y+z)^2$ .  $\square$

✓

Joshua Sawyer

Section 1.6, #6a,b

For the following statements:

$$x^2 = 4 \Rightarrow x = 2 \quad (4)$$

$$\text{If } 2x \leq x, \text{ then } x^2 > 0 \quad (5)$$

Find the converse, and determine the truth value for both the statement and its converse. Justify.

$$(1) \quad (P \Rightarrow Q) \quad x^2 = 4 \Rightarrow x = 2 \quad \text{F}$$

**Proof:** Let  $x = -2 \Rightarrow x^2 = 4 \Rightarrow x = 2$ . So, a contradiction is found.  
 $\therefore (P \Rightarrow Q)$  is F. ■

$$(Q \Rightarrow P) \quad x = 2 \Rightarrow x^2 = 4 \quad \text{T}$$

**Proof:** Let  $x = 2 \Rightarrow x^2 = 4$ .  
 $\therefore (Q \Rightarrow P)$  is T. ■

$$(2) \quad (P \Rightarrow Q) \quad \text{If } 2x \leq x, \text{ then } x^2 > 0 \quad \text{F}$$

**Proof:** Let  $x = 0$ .  $2 \cdot 0 \leq 0$ , but  $0^2 = 0 > 0$  is false. So, a contradiction is found.  
 $\therefore (P \Rightarrow Q)$  is F. ■

$$(Q \Rightarrow P) \quad \text{If } x^2 > 0, \text{ then } 2x \leq x \quad \text{F}$$

**Proof:** Let  $x = 2 \Rightarrow 4 > 0 \Rightarrow 4 \leq 2$ . But  $4 > 2$ , so a contradiction is found.  
 $\therefore (Q \Rightarrow P)$  is F. ■

Melissa Brown

Section 2.1, number 7

Let  $\ell$  be the line  $y = m(x-a) + a^2$  through the point  $(a, a^2)$  with slope  $m$ . Prove that  $\ell$  intersects the parabola  $y = x^2$  in exactly one point if and only if  $m = 2a$ .

The proof of this example involves two cases. The first case deals with the statement, if  $\ell$  intersects the parabola  $y = x^2$  in exactly one point then  $m = 2a$ . Assume  $m(x-a) + a^2 = x^2$   
 $x^2 - mx + (ma - a^2) = 0$ . So,  $x = (m \pm \sqrt{m^2 - 4(ma - a^2)})/2$ . And so,  $\sqrt{m^2 - 4(ma - a^2)} = 0$ . Now we have that  $m^2 - 4ma + 4a^2 = 0$  and  $m = (4a \pm \sqrt{16a^2 - 4(4a^2)})/2 = 2a$ . Therefore if  $\ell$  intersects  $y = x^2$  in one place if  $m = 2a$ . The second case deals with the statement, if  $m = 2a$  then  $\ell$  intersects the parabola  $y = x^2$  in exactly one point. The line and the parabola intersect when the two equations are equal. In other words the two equations intersect where  $x^2 = 2a(x-a) + a^2$ . Factoring we get that  $x^2 = 2ax - 2a^2 + a^2$ . Combining like terms you get  $x^2 - 2ax + a^2$ . Bringing all the variables to one side you get that  $x^2 - 2ax + a^2 = 0$ . In other words  $(x-a)^2 = 0$  or  $x = a$ . Therefore if  $m = 2a$ , then the line intersects the parabola in exactly one point.

✓

Ben Salisbury

Section 2.2 #2b

Prove  $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4} \quad \forall n \in \mathbb{N}$ .

Let  $n = 1$  LHS =  $\sum_0^1 i^3 = 0^3 + 1^3 = 1$  RHS =  $\frac{1^2(1+1)^2}{4} = 1$

Assume true for all  $k \geq 1$   $\sum_{i=0}^k i^3 = \frac{k^2(k+1)^2}{4}$

Show for  $k+1$

$$\begin{aligned} \sum_0^{k+1} i^3 &= \sum_0^k i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)(k+1)^2 \\ &= \frac{k^2(k+1)^2 + 4k + 4(k+1)^2}{4} \\ &= \frac{(k^2 + 4k + 4)(k+1)^2}{4} \\ &= \frac{(k+2)^2(k+1)^2}{4} \end{aligned}$$



Sara Day


Section 2.2. #4

Show that the sum of the cubes of any three consecutive natural numbers is a multiple of 9.

**Proof**  $n^3 + (n+1)^3 + (n+2)^3 = 9m$  where  $n, m \in \mathbb{N}$ .

Let  $n = 1$ , then  $1^3 + 2^3 + 3^3 = 36$  is divisible by 9 ( $m = 4$ ).

Assume that  $k^3 + (k+1)^3 + (k+2)^3 = 9m$  for  $k \geq 1$  and  $m \in \mathbb{N}$ . then we want to show  $(k+1)^3 + (k+2)^3 + (k+3)^3 = 9j$  for some  $j \in \mathbb{N}$

$$\begin{aligned}(k+1)^3 + (k+2)^3 + (k+3)^3 &= k^3 + (k+1)^3 + (k+2)^3 + (k+3)^3 - k^3 \\ &= 9m + (k+3)^3 - k^3 \\ &= 9m + k^3 + 9k^2 + 27k + 27 - k^3 \\ &= 9m + 9(k^2 + 3k + 1) \\ &= 9(m + k^2 + 3k + 1) \\ &= 9j(j = m + k^2 + 3k + 1) \blacksquare\end{aligned}$$


Ilana Marcus

Section 3.1, #20

Prove that  $4 \mid 13^n - 1 \forall n \in \mathbb{N}$ .

Using Mathematical Induction:

Base case:  $n=1$ . Then  $13^1 - 1 = 12$ ,  $12 = 4 \cdot 3$ , so the base case checks.

Induction Assumption: for  $k \geq 1$ ,  $4 \mid 13^k - 1$ . This means that for some  $m \in \mathbb{Z}$ ,  $13^k - 1 = 4m$ .

We want to show that  $4 \mid 13^{k+1} - 1$ , i.e.  $13^{k+1} - 1 = 4p$  for some  $p \in \mathbb{Z}$ .

We know  $13^{k+1} - 1 = (13^k \cdot 13) - 1$  from the exponent laws,

$$= [(4m+1) \cdot 13] - 1 \text{ from the induction assumption,}$$

$$= (52m+13) - 1 \text{ after distributing 13,}$$

$$= 52m + 12 \text{ after subtraction}$$

$$= 4(13m+3) \text{ after factoring out a 4.}$$

Since  $m \in \mathbb{Z}$ ,  $(13m+3) \in \mathbb{Z}$ . Therefore,  $13^{k+1} - 1 = 4p$ , where  $p = 13m+3$ , and  $p \in \mathbb{Z}$ .

Therefore,  $4 \mid 13^{k+1} - 1$ . ♦



William Day

**4.3, #6** Observe that the solution of Example 4.3.6. almost always involved the sum of entries of the eleventh row of Pascal's triangle. Conjecture and prove a formula for the sum of the entries of the  $n$ -th row of Pascal's triangle.

**Note** By the definition of Pascal's triangle, the zeroth row is  $\{1\}$  and the first row is  $\{1, 1\}$ ; for  $n > 1$  and  $0 < i < n$ , the  $i$ -th element of the  $n$ -th row is equal to the sum of the  $(i - 1)$ -th and  $i$ -th elements of the  $(n - 1)$ -th row and the first and last elements are always 1. Hence the first few rows are

$$\begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & 1 & 1 \\ & & & & & 1 & 2 & 1 \\ & & & 1 & 3 & 3 & 1 \\ & 1 & 4 & 6 & 4 & 1 \end{array}$$

**Lemma** For  $n > 1$  the  $n$ -th row of Pascal's triangle has the form

$$1 \binom{n}{1} \binom{n}{2} \cdots \binom{n}{n-1} 1.$$

**Proof** Let  $S_k$  be the statement that for  $k > 1$  the  $k$ -th row of Pascal's triangle has the form

$$1 \binom{k}{1} \binom{k}{2} \cdots \binom{k}{k-1} 1.$$

For  $k = 2$ ,  $S_k$  is the statement that the  $k$ -th row of Pascal's triangle has the form

$$1 \binom{2}{1} 1 = 1 \ 2 \ 1$$

is true. For some  $k \geq 1$ , assume  $S_k$  is true. By the definition of Pascal's triangle, the  $(k + 1)$ -th row is

$$1 \left[ \binom{k}{0} + \binom{k}{1} \right] \left[ \binom{k}{1} + \binom{k}{2} \right] \cdots \left[ \binom{k}{k-1} + \binom{k}{k} \right] 1.$$

Since  $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ , we see that  $S_{k+1}$

$$1 \binom{k+1}{1} \binom{k+1}{2} \cdots \binom{k+1}{k} 1$$

is true. Therefore, by the Principle of Mathematical Induction,  $S_n$  is true for all  $n \in \mathbb{N}$ .

**Conjecture** The sum of the entries in the  $n$ -th row of Pascal's triangle is  $2^n$ .

**Proof** By the previous Lemma and the identities  $\binom{n}{0} = \binom{n}{n} = 1$ , the sum of the entries in the  $n$ -th row of Pascal's triangle is

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{i=0}^n \binom{n}{i}.$$

By the Binomial Theorem which states that  $(x + y)^n = \sum_{j=0}^n \binom{n}{j} x^j y^{n-j}$ , and letting  $x = y = 1$ , we have

$$\sum_{i=0}^n \binom{n}{i} = (1 + 1)^n = 2^n. \blacksquare$$